Universal algebra and lattice theory
Week 1
Introduction and background

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Introduction

- Universal algebra is the study of algebraic structures.
- There are three questions you may now have:
  1. What is an algebraic structure?
  2. What precisely do we study about algebraic structures?
  3. Why should we do this?
- I will explain how I came to study this subject in order to answer these questions.

Introduction

- When I was about 12 I become interested in division by zero.
- Having seen the introduction of new symbols to solve equations like $2 + x = 1$, $3x = 1$, or $x^2 = -1$, I wanted to produce a new symbol, call it α, such that $0\alpha = 1$.
- Doing arithmetic with α seemed challenging, since many of the usual rules I knew didn't work any more.
- For example, we have that

$$(0 \cdot 0)\alpha = 0\alpha = 1$$

while

$$0(0\alpha) = 0 \cdot 1 = 0,$$

so the associative law must fail, for I do not want $1 = 0$.

Introduction

- Seven years later I was a student at Monroe Community College and I was a bit more mathematically sophisticated.
- I wrote a paper (laboriously, using the equation editor in Microsoft Word) in which I defined a collection of *apportional numbers* $\mathbb{A}$ as equivalence classes in analogy with the definition of the rational numbers.
- I defined a multiplication operation on these classes and proved that it was well-defined.
- My system failed to be associative, but I was also able to prove that no such system could satisfy the associative law.

Introduction

- I did an independent study in abstract algebra at the end of my time at community college, which covered group theory up to the Fundamental Theorem of Finite Abelian Groups.
- Inspired by my interest in division by zero, and now working a bit more abstractly, I defined a new class of objects to study.
- I said that a *ripple* is a set A equipped with a binary operation $\cdot$ such that for each $x \in A$ there exists some $y \in A$ such that $x \cdot y = y \cdot x = x$. (That is, x *absorbs* y.) There were no other assumptions on the nature of the binary operation.
- I was able to define direct products and isomorphisms for ripples and I proved a few results about them.

Introduction

- After community college I transferred to the University of Rochester.
- While an undergraduate there I learned about rings and modules in algebra courses, as well as their respective definitions of homomorphisms, products, and Isomorphism Theorems, which were all quite similar to those for groups.
- I distinctly remember being told on at least one occasion that we weren't going to prove one of the Isomorphism Theorems for a new class of algebraic structures because the proof was basically the same as the one we had done before.

Introduction

An example of a very similar pair of theorems is the following.

Theorem

Given a group homomorphism $h\colon G_1 \to G_2$ *we have that* $G_1/\ker(h) \cong h(G_1)$. *This isomorphism is given by* $g\ker(h) \mapsto h(g)$ *for* $g \in G_1$.

Theorem

Given a ring homomorphism $h\colon R_1 \to R_2$ *we have that* $R_1/\ker(h) \cong h(R_1)$. *This isomorphism is given by* $r + \ker(h) \mapsto h(r)$ *for* $r \in R_1$.

Introduction

- It seemed to me at this point logical that since groups, rings, and modules were all «algebraic structures» that there should be some general definition of an «algebra» with corresponding homomorphisms, direct products, quotients, and so forth which perhaps also included my own peculiar, not-necessarily-associative, systems.
- With such a setup one could hope to prove the Isomorphism Theorems once and for all, without making reference to the particular type of system under consideration.
- By around 2015 I had written down some definitions which were very similar to the following ones, which are standard.

Operations

- All of the algebraic structures we have seen (groups, rings, modules, even my oddball examples) have a collection of elements as well as ways to «add» or «multiply» them.
- In a group G we can think of the product of two elements as given by a function $f\colon G \times G \to G$. We can also think of the sum and product in a ring in this way.
- While most «multiplications» we see combine two elements to produce one other one, nonbinary «multiplications» also exist. These may combine three or more elements at once to produce a new element. We would like to treat all of these in the same way.

Operations

- We write $\mathbb{W} := \{0, 1, 2, \ldots\}$ for the set of *whole numbers*. Given a set A and some $n \in \mathbb{W}$ we define the collection of n-*tuples* in A to be

$$A^n := \{\, (a_1, \ldots, a_n) \mid a_1, \ldots, a_n \in A \,\}\,.$$

- We can also think of A^n as the set of all functions from $\{1, 2, \ldots, n\}$ to A.
- The set A^n itself is called the n^{th} *Cartesian power* of A.

Operations

- We describe A^n for small values of n.
- We have that A^0 contains a single element. We can think of this element as the *empty tuple* $()$ or as the *empty function* $e\colon \varnothing \to A$.
- We have that $A^1 \cong A$ since we can identify the 1-tuple (a) with the element a for any $a \in A$.
- Elements of A^2 are ordered pairs (a_1, a_2) for $a_1, a_2 \in A$
- Elements of A^3 are ordered triples (a_1, a_2, a_3) for $a_1, a_2, a_3 \in A$.

Operations

Definition (Operation, arity)

Given a set A and some $n \in \mathbb{W}$ we refer to a function $f\colon A^n \to A$ as an *n-ary operation* on A. When f is an n-ary operation on A we say that f has *arity* n.

- The case we're most familiar with is $n = 2$, which is that of a binary operation.
- An operation of arity 0 amounts to choosing an element of A. We call such operations *constant* or *nullary*.
- An operation of arity 1 is essentially a function from A to itself. We call such operations *unary*.
- Operations of arity 3 are ways of «multiplying» exactly three objects in a specified order to obtain another of the same type. We call such operations *ternary*.

Algebras

Definition (Algebra)

An *algebra* (A, F) consists of a set A and a sequence $F = \{f_i\}_{i \in I}$ of operations on A, indexed by some set I.

- We often write $\mathbf{A} := (A, F)$ to indicate that we're defining $\mathbf{A}$ to be the algebra (A, F).
- We refer to A as the *universe* of $\mathbf{A}$.
- We refer to the f_i as the *basic operations* of $\mathbf{A}$.
- We've now answered the question «What is an algebraic structure?».

Algebras

- We write $\mathbb{N} := \{1, 2, \dots\}$ for the set of *natural numbers*.
- We often consider algebras $\mathbf{A} := (A, F = \{f_i\}_{i \in I})$ where $I = \{1, 2, \dots, k\}$ for some $k \in \mathbb{N}$.
- In this case we write $\mathbf{A} := (A, f_1, f_2, \dots, f_k)$ rather than $\mathbf{A} := (A, F)$.
- Groups can be thought of as algebras of the form $\mathbf{G} := (G, *)$ where G is the set of elements of the group and $*\colon G^2 \to G$ is the group multiplication.
- Rings can similarly be thought of as algebras of the form $\mathbf{R} := (R, +, \cdot)$ where R is the set of elements of the ring and $+$ and $\cdot$ are the binary addition and multiplication operations, respectively.

Similarity types

- Note that groups as formulated previously have only a single binary operation, while rings have two binary operations.
- To get started we only want to compare algebras whose operations can be identified with each other.
- Given an algebra $\mathbf{A} := (A, \{f_i\}_{i\in I})$ we define a map $\rho\colon I \to \mathbb{W}$ where $\rho(i) := n$ when $f_i\colon A^n \to A$ is an n-ary operation on A.
- This map $\rho\colon I \to \mathbb{W}$ is called the *similarity type* of $\mathbf{A}$.
- When two algebras $\mathbf{A} := (A, F)$ and $\mathbf{B} := (B, G)$ have the same similarity type $\rho\colon I \to \mathbb{W}$ we say that $\mathbf{A}$ and $\mathbf{B}$ are *similar algebras*.

Subalgebras and homomorphisms

Given a set A we denote by $\mathrm{Sb}(A)$ the powerset of A, or the collection of all subsets of A.

Definition (Subalgebra, homomorphism)

Given algebras $\mathbf{A} := (A, F)$ and $\mathbf{B} := (B, G)$ of the same similarity type $\rho\colon I \to \mathbb{W}$ we say that

1. $\mathbf{B}$ is a *subalgebra* of $\mathbf{A}$ when $B \subset A$ and for each $i \in I$ we have that $g_i = f_i|_{B^{\rho(i)}}$ and
2. a function $h\colon A \to B$ is a *homomorphism* from $\mathbf{A}$ to $\mathbf{B}$ when for each $i \in I$ and all $a_1, \ldots, a_{\rho(i)} \in A$ we have that

$$h(f_i(a_1, \ldots, a_{\rho(i)})) = g_i(h(a_1), \ldots, h(a_{\rho(i)})).$$

Products

Direct (or external) products also generalize to arbitrary algebras.

Definition (Product)

Given a sequence $\left\{\mathbf{A}_j := (A_j, \{f_{i,j}\}_{i\in I})\right\}_{j\in J}$ of algebras of the same similarity type $\rho\colon I \to \mathbb{W}$ we define the *product* of $\{\mathbf{A}_j\}$ to be

$$\prod_{j\in J} \mathbf{A}_j := (A, \{g_i\}_{i\in I})$$

where $A := \prod_{j\in J} A_j$ and $g_i\colon A^{\rho(i)} \to A$ is given by

$$g_i(\{a_{1,j}\}_{j\in J}, \ldots, \{a_{\rho(i),j}\}_{j\in J}) := \{f_{i,j}(a_{1,j}, \ldots, a_{\rho(i),j})\}_{j\in J}\,.$$

Varieties

- Some people say that universal algebra is the study of *varieties*, which are classes of similar algebras closed with respect to homomorphisms, subalgebras, and products.
- This is an answer to the question «What precisely do we study about algebraic structures?», although it is a little too restrictive in my opinion.
- I was initially unsure that universal algebra captured what I wanted it to because I originally found that description of the discipline.
- I wanted something more down to earth to start with. Namely, I wanted to have a set of tools so that if I was given, for example, some crazy algebra $\mathbf{A} := (A, f_1, f_2)$ where $f_1 \colon A^{12} \to A$ and $f_2 \colon A^{313} \to A$ I would be able to «understand the structure» of that algebra or «decompose» it like I was able to do with groups and rings.

Motivation

- We finally address the question «Why should we do this?».
- First of all, I mainly wanted to understand the structure of general algebras because I thought it would be cool. Happily, universal algebra does provide tools for doing this.
- Like any mathematical subject, people often study universal algebra purely because they like the objects involved. Algebras themselves play the role that numbers do in number theory and later on we will see how to find special ones which are analogous to the prime numbers.
- Universal algebra has interesting ties to many areas of mathematics, including a very strong connection to lattice theory, which I will begin to describe next time.

Motivation

- Those connections will in some cases make themselves apparent as one learns the basic tools and examples. We will discuss others at the end of the semester. Some areas that touch universal algebra include graph theory, analysis, topology, and number theory. In pure algebra a great deal of work has been done on varieties of groups.
- The unified formulation of theorems and concepts in universal algebra can make working with the objects of classical abstract algebra easier. One gets a clearer idea of which properties actually depend on the particular class of algebras in question.
- Universal algebra also has applications in computer science.

Thank you!