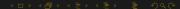
Math 2130 Linear Algebra Week 10 Homomorphisms

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Today's topics

Homomorphisms

- The function $f: \mathbb{R}^2 \to \mathbb{R}^2$ given by f(x, y) = (x, 0) is additive and homogeneous, but is not injective or surjective.
- The function $g: \mathbb{R}^2 \to \mathbb{R}^2$ given by g(x,y) = (x+y,2x+2y) is additive and homogeneous, but is not injective or surjective.
- More generally, note that if A is an $m \times n$ matrix and u and v are n-column vectors then A(u+v) = Au + Av.
- Also, if k is a scalar then A(ku) = k(Au).

 $lue{}$ Similarly, for any differentiable functions f and g we have

$$\frac{\mathrm{d}}{\mathrm{d}x}(f+g) = \frac{\mathrm{d}f}{\mathrm{d}x} + \frac{\mathrm{d}g}{\mathrm{d}x}$$

and

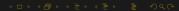
$$\frac{\mathrm{d}}{\mathrm{d}x}(kf) = k\frac{\mathrm{d}f}{\mathrm{d}x}.$$

■ We will now give a common generalization of these situations so we can treat them both with the same techniques.

Definition

Given vector spaces V and W a function $T\colon V\to W$ is said to be a homomorphism when for all $v_1,v_2\in V$ and all scalars k we have that

- \blacksquare (additivity) $T(v_1+v_2)=T(v_1)+T(v_2)$ and
- $(homogeneity) T(kv_1) = kT(v_1).$
- Isomorphisms are homomorphisms that are also bijections.
- Homomorphisms are also called *linear transformations* or *linear maps*.



- I will now verify that the following are homomorphisms:
 - **1** The function $f: \mathbb{R}^2 \to \mathbb{R}$ given by f(x,y) = ax + by for some fixed $a, b \in \mathbb{R}$.
 - The zero homomorphism $z \colon V \to W$ given by z(v) = 0 for any $v \in V$.
 - The map $T_A: \mathbb{R}^n \to \mathbb{R}^m$ given by $T_A(v) = Av$ where $A \in \operatorname{Mat}_{m \times n}(\mathbb{R})$.
 - If the projection $\pi_i: \mathbb{R}^n \to \mathbb{R}$ given by $\pi_i(x_1, \dots, x_n) = x_i$.
 - 5 The inclusion $\alpha_i : \mathbb{R} \to \mathbb{R}^n$ where $\alpha_i(x) = xe_i$ where e_i is the standard basis vector.