Math 2130
Linear Algebra
Week 10
Homomorphisms

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Today's topics

1. Homomorphisms

Homomorphisms

- The function $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ given by $f(x, y) = (x, 0)$ is additive and homogeneous, but is not injective or surjective.
- The function $g\colon \mathbb{R}^2 \to \mathbb{R}^2$ given by $g(x, y) = (x + y, 2x + 2y)$ is additive and homogeneous, but is not injective or surjective.
- More generally, note that if A is an $m \times n$ matrix and u and v are n-column vectors then $A(u + v) = Au + Av$.
- Also, if k is a scalar then $A(ku) = k(Au)$.

Homomorphisms

- Similarly, for any differentiable functions f and g we have

$$\frac{\mathrm{d}}{\mathrm{d}x}(f+g) = \frac{\mathrm{d}f}{\mathrm{d}x} + \frac{\mathrm{d}g}{\mathrm{d}x}$$

  and

$$\frac{\mathrm{d}}{\mathrm{d}x}(kf) = k\frac{\mathrm{d}f}{\mathrm{d}x}.$$

- We will now give a common generalization of these situations so we can treat them both with the same techniques.

Homomorphisms

Definition

Given vector spaces V and W a function $T\colon V \to W$ is said to be a *homomorphism* when for all $v_1, v_2 \in V$ and all scalars k we have that

1. (additivity) $T(v_1 + v_2) = T(v_1) + T(v_2)$ and
2. (homogeneity) $T(kv_1) = kT(v_1)$.

- Isomorphisms are homomorphisms that are also bijections.
- Homomorphisms are also called *linear transformations* or *linear maps*.

Homomorphisms

- I will now verify that the following are homomorphisms:
  1. The function $f\colon \mathbb{R}^2 \to \mathbb{R}$ given by $f(x, y) = ax + by$ for some fixed $a, b \in \mathbb{R}$.
  2. The zero homomorphism $z\colon V \to W$ given by $z(v) = 0$ for any $v \in V$.
  3. The map $T_A\colon \mathbb{R}^n \to \mathbb{R}^m$ given by $T_A(v) = Av$ where $A \in \mathrm{Mat}_{m\times n}(\mathbb{R})$.
  4. The projection $\pi_i\colon \mathbb{R}^n \to \mathbb{R}$ given by $\pi_i(x_1, \ldots, x_n) = x_i$.
  5. The inclusion $\alpha_i\colon \mathbb{R} \to \mathbb{R}^n$ where $\alpha_i(x) = xe_i$ where e_i is the standard basis vector.