Math 2130
Linear Algebra
Week 2
Gauss's method and matrices

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Today's topics

1. Matrices

Gauss's method

- Small systems of linear equations may be solved by substitution, but this is difficult (or impossible) for larger systems.
- There are, however, three basic ways we can change our view of a system of linear equations which can help us find solutions:
  1. Swap two equations.
  2. Add a nonzero multiple of one equation to another one.
  3. Multiply an equation by a nonzero scalar.

Matrices and linear systems

Definition

An $m \times n$ *matrix* is a rectangular array of numbers consisting of m rows and n columns.

$$A = [a_{i,j}] = \begin{bmatrix} 2 & 3 \\ i & e^2 \\ \frac{1}{2} & -3 \end{bmatrix}$$

- Matrices are often denoted by capital letters such as A, B, C, or M.
- The numbers in a matrix are called the *entries* or *components* of the matrix.

Matrices and linear systems

Definition

An $m \times n$ *matrix* is a rectangular array of numbers consisting of m rows and n columns.

$$A = [a_{i,j}] = \begin{bmatrix} 2 & 3 \\ i & e^2 \\ \frac{1}{2} & -3 \end{bmatrix}$$

- The *index notation* $A = [a_{i,j}]$ means the that entry in the i^{th} row and j^{th} column of A is labeled $a_{i,j}$.
- For example, here $a_{1,2} = 3$ and $a_{3,1} = \frac{1}{2}$.

Matrices and linear systems

$$A = [a_{i,j}] = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{bmatrix}$$

- A general $m \times n$ matrix is given above in index notation.
- Often commas are omitted when the context is clear, so we can write a_{25} instead of $a_{2,5}$. Just be careful when there are too many digits, since we can't tell if a_{123} means $a_{1,23}$ or $a_{12,3}$.

Matrices and linear systems

$$A = [a_{i,j}] = \begin{bmatrix} 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \end{bmatrix}$$

- Index notation is useful for specifying matrices quickly.
- The 2×4 matrix above is $A = [a_{ij}]$ where $a_{ij} = i + j$.
- Can anyone give the entry $b_{3,7}$ of the 1000×2^{100} matrix $B = [b_{ij}]$ where $b_{ij} = 3^i - j$?

Matrices and linear systems

$$A = [a_{i,j}] = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{bmatrix}$$

- When a matrix A has m rows and n columns we say that the *size* of A is $m \times n$.
- Many operations can only be performed with matrices of an appropriate size.

Matrices and linear systems

Definition

To each linear system we assign two matrices:

- The *matrix of coefficients*

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}.$$

Matrices and linear systems

Definition

To each linear system we assign two matrices:

- The *augmented matrix*

$$A^{\#} = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{array}\right].$$

Matrices and linear systems

- Given an augmented matrix $A^{\#}$ we can recover the original system of equations by viewing each a_{ij} as the coefficient of x_j in the i^{th} equation of the system and by viewing each b_j as the constant for the j^{th} equation.

Matrices and linear systems

Definition

A $m \times n$ matrix is said to be a *row-echelon matrix* when

1. all rows consisting entirely of zeroes are at the bottom of the matrix,
2. the first nonzero entry in any nonzero row is a 1 (called the *leading* 1), and
3. the leading 1 of any row below the first row is to the right of the leading 1 of the row above it.