Math 2130
Linear Algebra
Week 9
Inverse matrices

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Today's topics

1. Inverse matrices

Inverse matrices

- Fix an $n \times n$ matrix A and suppose that there is an $n \times n$ matrix A^{-1} such that

$$AA^{-1} = I_n = A^{-1}A.$$

- Such a matrix A^{-1} is called an *inverse* A.
- If we put a linear system into matrix form as $Ax = b$ then having such a matrix A^{-1} would yield that $A^{-1}Ax = A^{-1}b$ and hence that $x = A^{-1}b$ is the unique solution to the system.
- Note that if B is also an inverse of A then

$$A^{-1} = I_n A^{-1} = (BA)A^{-1} = B(AA^{-1}) = BI_n = B$$

  so the inverse of A, if it exists, is unique.
- We say that A is *invertible* if A^{-1} exists.

Inverse matrices

- We can set up a system of linear equations to find the inverse of a 2×2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$.
- If
$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} u & v \\ x & y \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$
then we can solve the system
$$au + bx = 1,\ av + by = 0,\ cu + dx = 0, \text{ and } cv + dy = 1$$
for x, y, z, and w.

Inverse matrices

- We get that

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

- A 2×2 matrix is invertible only when this formula makes sense. That is, when $ad - bc \neq 0$.

Inverse matrices

- We can use this formula to see that

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

and that

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Inverse matrices

- We can do the same method for 3×3 matrices, but there is a better way to do find inverses that producing the general formula.

Inverse matrices

- Gauss-Jordan elimination can be used to find the inverse of a matrix.
- We can represent the elementary row operations as *elementary matrices*, all of which are invertible.