# Math 2130
# Linear Algebra
# Week 9
# Inverse matrices

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2025 October 22

# Today’s topics

1. Inverse matrices

# Inverse matrices

- Gauss-Jordan elimination can be used to find the inverse of a matrix.
- We can represent the elementary row operations as *elementary matrices*, all of which are invertible.

# Inverse matrices

- In order to get the elementary matrix for a row operation, apply that operation to the identity.
- For example, when there are two rows we can represent adding $3$ times row two to row one by

$$\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}.$$

# Inverse matrices

- If we multiply this by another matrix on the right, it has the effect of adding three times the second row of that matrix to the first.
- Observe that

$$\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 10 & 14 \\ 3 & 4 \end{bmatrix}.$$

# Inverse matrices

- Swapping the first two rows when there are three total rows is represented by
$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

# Inverse matrices

- Multiplying the second row by $5$ when there are three total rows is represented by

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

# Inverse matrices

- If an $n \times n$ matrix $A$ can be reduced to the identity $I_n$ then we have a sequence of elementary matrices $E_1, \ldots, E_k$ such that

$$E_k E_{k-1} \cdots E_2 E_1 A = I_n.$$

- This means that $E_k E_{k-1} \cdots E_2 E_1 = A^{-1}$.

# Inverse matrices

- Notice that $E_k E_{k-1} \cdots E_2 E_1$ can be evaluated by applying the elementary row operations corresponding to $E_2, E_3, \ldots, E_k$ to $E_1$ in the appropriate order.
- An even nicer method comes from noticing that $E_k E_{k-1} \cdots E_2 E_1 = E_k E_{k-1} \cdots E_2 E_1 I_n$, so $E_k E_{k-1} \cdots E_2 E_1$ can be evaluated by applying the elementary row operations corresponding to $E_1, E_2, \ldots, E_k$ to $I_n$ in the appropriate order.

# Inverse matrices

- From this it follows that we can start with a square matrix $A$ and make a new matrix $[A|I]$ with the same-sized identity.
- If we can row reduce this until it is of the form $[I|B]$ for some matrix $B$, it will turn out that $B = A^{-1}$.
- I'll illustrate this with the matrix $\begin{bmatrix} 0 & 2 \\ 1 & 3 \end{bmatrix}$.

# Inverse matrices

- Now let's consider a bigger example: $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix}$.

# Inverse matrices

- Note that we might get stuck and not be able to reduce our matrix to the identity.
- Consider the case of $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$.
- We saw before that this couldn't have an inverse because it made $ad - bc = 0$ in our formula for $2 \times 2$ matrices.
- If this matrix had an inverse then so would $\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$, but

$$\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a + 2c & c + 2d \\ 0 & 0 \end{bmatrix} \neq I_2.$$