Multiplayer rock-paper-scissors

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Introduction

- In the summer of 2017 I lived in a cave in Yosemite National Park.
- While there I wanted to explain to my friends that I study universal algebra.
- I realized that rock-paper-scissors was a particularly simple way to do that.

Introduction

We will view the game of RPS as a magma $\mathbf{A} := (A, f)$. We let $A := \{r, p, s\}$ and define a binary operation $f\colon A^2 \to A$ where $f(x, y)$ is the winning item among $\{x, y\}$.

$$\begin{array}{c|ccc} & r & p & s \\ \hline r & r & p & r \\ p & p & p & s \\ s & r & s & s \end{array}$$

Introduction

- I also realized that I wanted to be able to play with many of my friends at the same time.
- This led me to study hypertournaments and hypertournament algebras.

Talk outline

- Definition of RPS and PRPS magmas
- A numerical constraint relating arity and order
- Regular RPS magmas
- Hypertournaments
- An embedding result

Properties of RPS

The game RPS is

1. conservative,
2. essentially polyadic,
3. strongly fair, and
4. nondegenerate.

These are the properties we want for a multiplayer game, as well.

What does a multiplayer game mean?

- Suppose we have an *n-ary magma* $\mathbf{A} := (A, f)$ where $f\colon A^n \to A$.
- The *selection game* for $\mathbf{A}$ has n players, $p_1, p_2, \ldots, p_n$.
- Each player p_i simultaneously chooses an item $a_i \in A$.
- The winners of the game are all players who chose $f(a_1, \ldots, a_n)$.

Properties of RPS: conservativity

- We say that an operation $f\colon A^n \to A$ is *conservative* when for any $a_1, \ldots, a_n \in A$ we have that $f(a_1, \ldots, a_n) \in \{a_1, \ldots, a_n\}$.
- We say that $\mathbf{A}$ is conservative when each round has at least one winning player.

Properties of RPS: essential polyadicity

- We say that an operation $f\colon A^n \to A$ is *essentially polyadic* when there exists some $g\colon \mathrm{Sb}(A) \to A$ such that for any $a_1, \ldots, a_n \in A$ we have $f(a_1, \ldots, a_n) = g(\{a_1, \ldots, a_n\})$.
- We say that **A** is essentially polyadic when a round's winning item is determined solely by which items were played, not taking into account which player played which item or how many players chose a particular item (as long at it was chosen at least once).

Properties of RPS: strong fairness

- Let A_k denote the members of A^n which have k distinct components for some $k \in \mathbb{N}$.
- We say that f is *strongly fair* when for all $a, b \in A$ and all $k \in \mathbb{N}$ we have $\left|f^{-1}(a) \cap A_k\right| = \left|f^{-1}(b) \cap A_k\right|$.
- We say that **A** is strongly fair when each item has the same chance of being the winning item when exactly k distinct items are chosen for any $k \in \mathbb{N}$.

Properties of RPS: nondegeneracy

- We say that f is *nondegenerate* when $|A| > n$.
- In the case that $|A| \leq n$ we have that all members of $A_{|A|}$ have the same set of components.
- If $\mathbf{A}$ is essentially polyadic with $|A| \leq n$ it is impossible for $\mathbf{A}$ to be strongly fair unless $|A| = 1$.

Variants with more items

The French version of RPS adds one more item: the well. This game is not strongly fair but is conservative and essentially polyadic.

	r	p	s	w
r	r	p	r	w
p	p	p	s	p
s	r	s	s	w
w	w	p	w	w

Variants with more items

The recent variant Rock-Paper-Scissors-Spock-Lizard is conservative, essentially polyadic, strongly fair, and nondegenerate.

	r	p	s	v	l
r	r	p	r	v	r
p	p	p	s	p	l
s	r	s	s	v	s
v	v	p	v	v	l
l	r	l	s	l	l

Result for two-player games

The only "valid" RPS variants for two players use an odd number of items.

Proposition

Let $\mathbf{A}$ *be a selection game with* $n = 2$ *which is essentially polyadic, strongly fair, and nondegenerate and let* $m := |A|$. *We have that* $m \neq 1$ *is odd. Conversely, for each odd* $m \neq 1$ *there exists such a selection game.*

Proof.

We need $m \mid \binom{m}{2}$. $\square$

PRPS magmas

Definition (PRPS magma)

Let $\mathbf{A} := (A, f)$ be an n-ary magma. When $\mathbf{A}$ is essentially polyadic, strongly fair, and nondegenerate we say that $\mathbf{A}$ is a PRPS *magma* (read "*pseudo*-RPS *magma*"). When $\mathbf{A}$ is an n-magma of order $m \in \mathbb{N}$ with these properties we say that $\mathbf{A}$ is a PRPS(m, n) *magma*. We also use PRPS and PRPS(m, n) to indicate the classes of such magmas.

Result for multiplayer games

Theorem

Let $\mathbf{A} \in \mathrm{PRPS}(m, n)$ *and let* $\varpi(m)$ *denote the least prime dividing* m*. We have that* $n < \varpi(m)$*. Conversely, for each pair* (m, n) *with* $m \neq 1$ *such that* $n < \varpi(m)$ *there exists such a magma.*

Proof.

We need $m \mid \gcd\left(\left\{\binom{m}{2}, \ldots, \binom{m}{n}\right\}\right)$. $\square$

RPS magmas

Definition (RPS magma)

Let $\mathbf{A} := (A, f)$ be an n-ary magma. When $\mathbf{A}$ is conservative, essentially polyadic, strongly fair, and nondegenerate we say that $\mathbf{A}$ is an RPS *magma*. When $\mathbf{A}$ is an n-magma of order m with these properties we say that $\mathbf{A}$ is an RPS(m, n) *magma*. We also use RPS and RPS(m, n) to indicate the classes of such magmas.

Both the original game of rock-paper-scissors and the game rock-paper-scissors-Spock-lizard are RPS magmas. The French variant of rock-paper-scissors is not even a PRPS magma.

A game for three players

- We now show how to construct a game for three players.
- This will be a ternary RPS magma $(A, f \colon A^3 \to A)$.
- Since $n = 3$ in this case and we require that $n < \varpi(m)$ we must have that $|A| \geq 5$.
- Our construction will use the left-addition action of $\mathbb{Z}_5$ on itself.
- We will produce an operation $f \colon \mathbb{Z}_5^3 \to \mathbb{Z}_5$ which is essentially polyadic with $w + f(x, y, z) = f(w + x, w + y, w + z)$ for any $w \in \mathbb{Z}_5$.
- Thus, we need only define f on a representative of each orbit of $\binom{\mathbb{Z}_5}{1}$, $\binom{\mathbb{Z}_5}{2}$, and $\binom{\mathbb{Z}_5}{3}$ under this action of $\mathbb{Z}_5$.

A game for three players

First we list the orbits of $\binom{\mathbb{Z}_5}{1}$, $\binom{\mathbb{Z}_5}{2}$, and $\binom{\mathbb{Z}_5}{3}$ under this action of $\mathbb{Z}_5$.

0	01	02	012	013
1	12	13	123	124
2	23	24	234	230
3	34	30	340	341
4	40	41	401	402

A game for three players

Next, we choose a representative for each orbit, say the first one in each row of this table.

0	01	02	012	013
1	12	13	123	124
2	23	24	234	230
3	34	30	340	341
4	40	41	401	402

A game for three players

Choose from each representative a particular element. For example, if our representative is 013 we may choose 0 as our special element. We also could have chosen 1 or 3, but not 2 or 4.

$0 \mapsto 0$	$01 \mapsto 1$	$02 \mapsto 0$	$012 \mapsto 0$	$013 \mapsto 0$
1	12	13	123	124
2	23	24	234	230
3	34	30	340	341
4	40	41	401	402

A game for three players

Use the left-addition action of $\mathbb{Z}_5$ to extend these choices to all members of $\binom{\mathbb{Z}_5}{1}$, $\binom{\mathbb{Z}_5}{2}$, and $\binom{\mathbb{Z}_5}{3}$.

$0 \mapsto 0$	$01 \mapsto 1$	$02 \mapsto 0$	$012 \mapsto 0$	$013 \mapsto 0$
$1 \mapsto 1$	$12 \mapsto 2$	$13 \mapsto 1$	$123 \mapsto 1$	$124 \mapsto 1$
$2 \mapsto 2$	$23 \mapsto 3$	$24 \mapsto 2$	$234 \mapsto 2$	$230 \mapsto 2$
$3 \mapsto 3$	$34 \mapsto 4$	$30 \mapsto 3$	$340 \mapsto 3$	$341 \mapsto 3$
$4 \mapsto 4$	$40 \mapsto 0$	$41 \mapsto 4$	$401 \mapsto 4$	$402 \mapsto 4$

A game for three players

We can read off a definition for the operation $f\colon \mathbb{Z}_5^3 \to \mathbb{Z}_5$ from this table. For example, we take $24 \mapsto 2$ to indicate that

$$f(2,4,4) = f(4,2,4) = f(4,4,2) = f(4,2,2) = f(2,4,2) = f(2,2,4) = 2.$$

$0 \mapsto 0$	$01 \mapsto 1$	$02 \mapsto 0$	$012 \mapsto 0$	$013 \mapsto 0$
$1 \mapsto 1$	$12 \mapsto 2$	$13 \mapsto 1$	$123 \mapsto 1$	$124 \mapsto 1$
$2 \mapsto 2$	$23 \mapsto 3$	$24 \mapsto 2$	$234 \mapsto 2$	$230 \mapsto 2$
$3 \mapsto 3$	$34 \mapsto 4$	$30 \mapsto 3$	$340 \mapsto 3$	$341 \mapsto 3$
$4 \mapsto 4$	$40 \mapsto 0$	$41 \mapsto 4$	$401 \mapsto 4$	$402 \mapsto 4$

A game for three players

The Cayley table for the 3-magma $\mathbf{A} := (\mathbb{Z}_5, f)$ obtained from this choice of f is given below.

0	0	1	2	3	4
0	0	1	0	3	0
1	1	1	0	0	4
2	0	0	0	2	4
3	3	0	2	3	3
4	0	4	4	3	0

1	0	1	2	3	4
0	1	1	0	0	4
1	1	1	2	1	4
2	0	2	2	1	1
3	0	1	1	1	3
4	4	4	1	3	4

2	0	1	2	3	4
0	0	0	0	2	4
1	0	2	2	1	1
2	0	2	2	3	2
3	2	1	3	3	2
4	4	1	2	2	2

3	0	1	2	3	4
0	3	0	2	3	3
1	0	1	1	1	3
2	2	1	3	3	2
3	3	1	3	3	4
4	3	3	2	4	4

4	0	1	2	3	4
0	0	4	4	3	0
1	4	4	1	3	4
2	4	1	2	2	2
3	3	3	2	4	4
4	0	4	2	4	4

α-action magmas

Definition (α-action magma)

Fix a group $\mathbf{G}$, a set A, and some $n < |A|$. Given a regular group action $\alpha\colon \mathbf{G} \to \mathbf{Perm}(A)$ such that each of the k-extensions of α is free for $1 \leq k \leq n$ let $\Psi_k := \left\{ \operatorname{Orb}(U) \;\middle|\; U \in \binom{A}{k} \right\}$ where $\operatorname{Orb}(U)$ is the orbit of U under α_k. Let $\beta := \{\beta_k\}_{1 \leq k \leq n}$ be a sequence of choice functions $\beta_k\colon \Psi_k \to \binom{A}{k}$ such that $\beta_k(\psi) \in \psi$ for each $\psi \in \Psi_k$. Let $\gamma := \{\gamma_k\}_{1 \leq k \leq n}$ be a sequence of functions $\gamma_k\colon \Psi_k \to A$ such that $\gamma_k(\psi) \in \beta_k(\psi)$ for each $\psi \in \Psi_k$. Let $g\colon \operatorname{Sb}_{\leq n}(A) \to A$ be given by $g(U) := (\alpha(s))(\gamma_k(\psi))$ when $U = (\alpha_k(s))(\beta_k(\psi))$. Define $f\colon A^n \to A$ by $f(a_1, \ldots, a_n) := g(\{a_1, \ldots, a_n\})$. The *$\alpha$-action magma* induced by (β, γ) is $\mathbf{A} := (A, f)$.

α-action magmas are RPS magmas

Theorem

Let $\mathbf{A}$ *be an* α*-action magma induced by* (β, γ)*. We have that* $\mathbf{A} \in \mathrm{RPS}$*.*

Definition (Regular RPS magma)

Let $\mathbf{G}$ be a nontrivial finite group and fix $n < \varpi(|G|)$. We denote by $\mathbf{G}_n(\beta, \gamma)$ the left-multiplication-action n-magma induced by (β, γ), which we refer to as a *regular* RPS *magma*.

Hypergraphs

Definition (Pointed hypergraph)

A *pointed hypergraph* $\mathbf{S} := (S, \sigma, g)$ consists of a hypergraph (S, σ) and a map $g\colon \sigma \to S$ such that for each edge $e \in \sigma$ we have that $g(e) \in e$. The map g is called a *pointing* of (S, σ).

Definition (n-complete hypergraph)

Given a set S we denote by $\mathbf{S}_n$ the *n-complete hypergraph* whose vertex set is S and whose edge set is $\bigcup_{k=1}^{n} \binom{S}{k}$.

Hypertournaments

Definition (Hypertournament)

An *n-hypertournament* is a pointed hypergraph $\mathbf{T} := (T, \tau, g)$ where $(T, \tau) = \mathbf{S}_n$ for some set S.

U	0	1	2	01	12	23	34	40	02	13	24	30	41
$g(U)$	0	1	2	1	2	3	4	0	0	1	2	3	4

U	012	123	234	340	401	013	124	230	341	402
$g(U)$	0	1	2	3	4	0	1	2	3	4

RPS$(5, 3)$ example

Hypertournament magmas

Definition (Hypertournament magma)

Given an n-hypertournament $\mathbf{T} := (T, \tau, g)$ the *hypertournament magma* obtained from $\mathbf{T}$ is the n-magma $\mathbf{A} := (T, f)$ where for $u_1, \ldots, u_n \in T$ we define

$$f(u_1, \ldots, u_n) := g(\{u_1, \ldots, u_n\}).$$

Definition (Hypertournament magma)

A *hypertournament magma* is an n-magma which is conservative and essentially polyadic.

Tournaments

- Tournaments are the $n = 2$ case of a hypertournament.
- Hedrlín and Chvátal introduced the $n = 2$ case of a hypertournament magma in 1965.
- There has been a lot of work on varieties generated by tournament magmas. See for example the survey by Crvenković et al. (1999).
- There are algebraic motivations for what follows, but I won't get into them now.

An embedding result

Definition (Regular balanced hypertournament)

We refer to a hypertournament $\mathbf{T} := (T, \tau, g)$ as a *regular balanced hypertournament* when the hypertournament magma of $\mathbf{T}$ is a regular RPS magma.

- It would be very nice if each finite n-hypertournament embedded into a finite regular balanced hypertournament.
- This turns out to be the case.

An embedding result

- Note that in a regular binary RPS magma $\mathbf{G}_2(\beta, \gamma)$ we have that

$$f(e, x) = xf(x^{-1}, e)$$

  so exactly one of $f(e, x) = e$ or $f(x^{-1}, e) = e$ holds.
- Note also that the orbit of $\{x, y\}$ contains $\left\{e, x^{-1}y\right\}$ and $y^{-1}x, e$, where $x^{-1}y$ and $y^{-1}x$ are inverses.
- We need then only define a map λ specifying for each pair of inverses $\left\{x, x^{-1}\right\}$ whether $f(e, x) = e$ or $f(e, x^{-1}) = e$ in order to specify $\mathbf{G}_2(\beta, \gamma)$.
- We can think of $\lambda(\left\{x, x^{-1}\right\})$ as choosing the «positive direction» with respect to x and x^{-1}.

An embedding result

In order to do this in general we need an n-ary analogue of inverses.

Definition (Obverse k-set)

Given $n > 1$, a nontrivial finite group $\mathbf{G}$ with $n < \varpi(|G|)$, $1 \leq k \leq n-1$, and $U, V \in \binom{G \setminus \{e\}}{k}$ we say that V is an *obverse* of U when $U = \{a_1, \ldots, a_k\}$ and there exists some $a_i \in U$ such that $V = \{a_i^{-1}\} \cup \{\, a_i^{-1} a_j \mid i \neq j \,\}$. We denote by $\mathrm{Obv}(U)$ the set consisting of all obverses V of U, as well as U itself.

The obverses of a set U are the nonidentity elements in the members of $\mathrm{Orb}(U \cup \{e\}) \setminus (U \cup \{e\})$ which contain e.

An embedding result

In order to specify $\mathbf{G}_n(\beta, \gamma)$ it suffices to choose the member $\{a_1, \dots, a_k\}$ of each collection of obverses for which $f(e, \dots, e, a_1, \dots, a_k) = e$.

Definition (*n*-sign function)

Given $n > 1$ and a nontrivial group $\mathbf{G}$ with $n < \varpi(|G|)$ let $\mathrm{Sgn}_n(\mathbf{G})$ denote the set of all choice functions on

$$\left\{ \mathrm{Obv}(U) \;\middle|\; (\exists k \in \{1, \dots, n-1\}) \left(U \in \binom{G \setminus \{e\}}{k} \right) \right\}.$$

We refer to a member $\lambda \in \mathrm{Sgn}_n(\mathbf{G})$ as an *n-sign function* on $\mathbf{G}$.

We then write $\mathbf{G}_n(\lambda)$ instead of $\mathbf{G}_n(\beta, \gamma)$.

An embedding result

- Now we can give an embedding of any finite hypertournament into a finite regular balanced hypertournament.
- Consider a finite hypertournament $\mathbf{T} := (T, \tau, g)$.
- Take $\mathbf{G} := \bigoplus_{u \in T} \mathbb{Z}_{\alpha_u}$ where $n < \varpi(\alpha_u)$ and $\mathbb{Z}_{\alpha_u} = \langle u \rangle$.
- We define an n-sign function $\lambda \in \mathrm{Sgn}_n(\mathbf{G})$.
- When $g(\{u_1, \ldots, u_k\}) = u_1$ we define

$$\lambda(\mathrm{Obv}(\{\, u_i - u_1 \mid i \neq 1 \,\})) := \{\, u_i - u_1 \mid i \neq 1 \,\}.$$

- Any values may be chosen for other orbits.
- The n-hypertournament corresponding to $\mathbf{G}_n(\lambda)$ contains a copy of $\mathbf{T}$.

An embedding result

- If we want a class of finite regular balanced hypertournaments in which any finite hypertournament embeds, we need only use magmas of the form $\mathbf{G}_n(\lambda)$ where:
  1. $\mathbf{G} = \mathbb{Z}^m_{\kappa(n)}$ where $\kappa(n)$ is the least prime strictly greater than n or
  2. $\mathbf{G} = \mathbb{Z}_{\alpha(m)}$ where $\alpha(m) := \prod_{k=\ell}^{m+\ell-1} p_k$ where p_k is the k^{th} prime and $\kappa(n) = p_\ell$.
- In particular, every tournament of order m embeds into the tournament corresponding to some regular RPS magma of the form $(\mathbb{Z}^m_3)_2(\lambda)$.
- This means that any tournament of order m embeds into some balanced tournament of order 3^m.

An embedding result

Definition (Balanced hypertournament)

We say that a hypertournament $\mathbf{T}$ is *balanced* when the hypertournament magma of $\mathbf{T}$ is an RPS magma.

- Let $h_n(m)$ denote the least natural such that each n-hypertournament of order m is contained in some balanced n-hypertournament of order $h_n(m)$.
- Our previous observation was that $h_2(m) \leq 3^m$.
- We can do much better than this.

An embedding result

Proposition

Every tournament of order m embeds into a balanced tournament of order $2m+1$.

(I drew a picture to prove this when I gave the talk.)

An embedding result

Proposition

Every tournament of order m embeds into a balanced tournament of order $2m+1$.

- One can show by example that $h_2(m) \geq 2m-1$.
- A similar construction to the one given previously shows that $h_2(m) \leq 2m-1$ so $h_2(m) = 2m-1$.
- I only produced the construction given for general hypertournaments once I found that I couldn't see how to generalize the doubling construction from the $n=2$ case.
- It would be interesting to know whether the argument generalizes.

An embedding result

- By the general embedding result we know that $h_n(m) \leq \kappa(n)^m$.
- For $n > 2$ is this the best bound possible?
- Are there some easy examples like in the $n = 2$ case which give a lower bound?

Thank you.