Multiplayer rock-paper-scissors

Charlotte Aten

University of Rochester

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Introduction

- In the summer of 2017 I lived in a cave in Yosemite National Park.
- While there I wanted to explain to my friends that I study universal algebra.
- I realized that rock-paper-scissors was a particularly simple way to do that.

Introduction

We will view the game of RPS as a magma $\mathbf{A} := (A, f)$. We let $A := \{r, p, s\}$ and define a binary operation $f\colon A^2 \to A$ where $f(x, y)$ is the winning item among $\{x, y\}$.

	r	p	s
r	r	p	r
p	p	p	s
s	r	s	s

Introduction

- I also realized that I wanted to be able to play with many of my friends at the same time.
- Naturally, this led me to study the varieties generated by hypertournament algebras.

Talk outline

- Definition of RPS and PRPS magmas
- A numerical constraint relating arity and order
- Regular RPS magmas
- Hypertournaments
- A generation result
- Automorphisms and congruences of regular RPS magmas
- The search for a basis of the variety generated by tournament algebras

Properties of RPS

The game RPS is

1. conservative,
2. essentially polyadic,
3. strongly fair, and
4. nondegenerate.

These are the properties we want for a multiplayer game, as well.

What does a multiplayer game mean?

- Suppose we have an *n-ary magma* $\mathbf{A} := (A, f)$ where $f\colon A^n \to A$.
- The *selection game* for $\mathbf{A}$ has n players, $p_1, p_2, \ldots, p_n$.
- Each player p_i simultaneously chooses an item $a_i \in A$.
- The winners of the game are all players who chose $f(a_1, \ldots, a_n)$.

Properties of RPS: conservativity

- We say that an operation $f\colon A^n \to A$ is *conservative* when for any $a_1, \ldots, a_n \in A$ we have that $f(a_1, \ldots, a_n) \in \{a_1, \ldots, a_n\}$.
- We say that $\mathbf{A}$ is conservative when each round has at least one winning player.

Properties of RPS: essential polyadicity

- We say that an operation $f\colon A^n \to A$ is *essentially polyadic* when there exists some $g\colon \mathrm{Sb}(A) \to A$ such that for any $a_1, \ldots, a_n \in A$ we have $f(a_1, \ldots, a_n) = g(\{a_1, \ldots, a_n\})$.
- We say that **A** is essentially polyadic when a round's winning item is determined solely by which items were played, not taking into account which player played which item or how many players chose a particular item (as long at it was chosen at least once).

Properties of RPS: strong fairness

- Let A_k denote the members of A^n which have k distinct components for some $k \in \mathbb{N}$.
- We say that f is *strongly fair* when for all $a, b \in A$ and all $k \in \mathbb{N}$ we have $\left|f^{-1}(a) \cap A_k\right| = \left|f^{-1}(b) \cap A_k\right|$.
- We say that **A** is strongly fair when each item has the same chance of being the winning item when exactly k distinct items are chosen for any $k \in \mathbb{N}$.

Properties of RPS: nondegeneracy

- We say that f is *nondegenerate* when $|A| > n$.
- In the case that $|A| \leq n$ we have that all members of $A_{|A|}$ have the same set of components.
- If $\mathbf{A}$ is essentially polyadic with $|A| \leq n$ it is impossible for $\mathbf{A}$ to be strongly fair unless $|A| = 1$.

Variants with more items

The French version of RPS adds one more item: the well. This game is not strongly fair but is conservative and essentially polyadic.

	r	p	s	w
r	r	p	r	w
p	p	p	s	p
s	r	s	s	w
w	w	p	w	w

Variants with more items

The recent variant Rock-Paper-Scissors-Spock-Lizard is conservative, essentially polyadic, strongly fair, and nondegenerate.

	r	p	s	v	l
r	r	p	r	v	r
p	p	p	s	p	l
s	r	s	s	v	s
v	v	p	v	v	l
l	r	l	s	l	l

Result for two-player games

The only “valid” RPS variants for two players use an odd number of items.

Proposition

Let $\mathbf{A}$ *be a selection game with* $n = 2$ *which is essentially polyadic, strongly fair, and nondegenerate and let* $m := |A|$. *We have that* $m \neq 1$ *is odd. Conversely, for each odd* $m \neq 1$ *there exists such a selection game.*

Proof.

We need $m \mid \binom{m}{2}$. □

PRPS magmas

Definition (PRPS magma)

Let $\mathbf{A} := (A, f)$ be an n-ary magma. When $\mathbf{A}$ is essentially polyadic, strongly fair, and nondegenerate we say that $\mathbf{A}$ is a PRPS *magma* (read "*pseudo*-RPS *magma*"). When $\mathbf{A}$ is an n-magma of order $m \in \mathbb{N}$ with these properties we say that $\mathbf{A}$ is a PRPS(m, n) *magma*. We also use PRPS and PRPS(m, n) to indicate the classes of such magmas.

Result for multiplayer games

Theorem

Let $\mathbf{A} \in \mathrm{PRPS}(m, n)$ *and let* $\varpi(m)$ *denote the least prime dividing* m*. We have that* $n < \varpi(m)$*. Conversely, for each pair* (m, n) *with* $m \neq 1$ *such that* $n < \varpi(m)$ *there exists such a magma.*

Proof.

We need $m \mid \gcd\left(\left\{\binom{m}{2}, \ldots, \binom{m}{n}\right\}\right)$. □

RPS magmas

Definition (RPS magma)

Let $\mathbf{A} := (A, f)$ be an n-ary magma. When $\mathbf{A}$ is conservative, essentially polyadic, strongly fair, and nondegenerate we say that $\mathbf{A}$ is an RPS *magma*. When $\mathbf{A}$ is an n-magma of order m with these properties we say that $\mathbf{A}$ is an RPS(m, n) *magma*. We also use RPS and RPS(m, n) to indicate the classes of such magmas.

Both the original game of rock-paper-scissors and the game rock-paper-scissors-Spock-lizard are RPS magmas. The French variant of rock-paper-scissors is not even a PRPS magma.

A game for three players

- We now show how to construct a game for three players.
- This will be a ternary RPS magma $(A, f\colon A^3 \to A)$.
- Since $n = 3$ in this case and we require that $n < \varpi(m)$ we must have that $|A| \geq 5$.
- Our construction will use the left-addition action of $\mathbb{Z}_5$ on itself.
- We will produce an operation $f\colon \mathbb{Z}_5^3 \to \mathbb{Z}_5$ which is essentially polyadic with $w + f(x, y, z) = f(w + x, w + y, w + z)$ for any $w \in \mathbb{Z}_5$.
- Thus, we need only define f on a representative of each orbit of $\binom{\mathbb{Z}_5}{1}$, $\binom{\mathbb{Z}_5}{2}$, and $\binom{\mathbb{Z}_5}{3}$ under this action of $\mathbb{Z}_5$.

A game for three players

First we list the orbits of $\binom{\mathbb{Z}_5}{1}$, $\binom{\mathbb{Z}_5}{2}$, and $\binom{\mathbb{Z}_5}{3}$ under this action of $\mathbb{Z}_5$.

0	01	02	012	013
1	12	13	123	124
2	23	24	234	230
3	34	30	340	341
4	40	41	401	402

A game for three players

Next, we choose a representative for each orbit, say the first one in each row of this table.

0	01	02	012	013
1	12	13	123	124
2	23	24	234	230
3	34	30	340	341
4	40	41	401	402

A game for three players

Choose from each representative a particular element. For example, if our representative is 013 we may choose 0 as our special element. We also could have chosen 1 or 3, but not 2 or 4.

$0 \mapsto 0$	$01 \mapsto 1$	$02 \mapsto 0$	$012 \mapsto 0$	$013 \mapsto 0$
1	12	13	123	124
2	23	24	234	230
3	34	30	340	341
4	40	41	401	402

A game for three players

Use the left-addition action of $\mathbb{Z}_5$ to extend these choices to all members of $\binom{\mathbb{Z}_5}{1}$, $\binom{\mathbb{Z}_5}{2}$, and $\binom{\mathbb{Z}_5}{3}$.

$0 \mapsto 0$	$01 \mapsto 1$	$02 \mapsto 0$	$012 \mapsto 0$	$013 \mapsto 0$
$1 \mapsto 1$	$12 \mapsto 2$	$13 \mapsto 1$	$123 \mapsto 1$	$124 \mapsto 1$
$2 \mapsto 2$	$23 \mapsto 3$	$24 \mapsto 2$	$234 \mapsto 2$	$230 \mapsto 2$
$3 \mapsto 3$	$34 \mapsto 4$	$30 \mapsto 3$	$340 \mapsto 3$	$341 \mapsto 3$
$4 \mapsto 4$	$40 \mapsto 0$	$41 \mapsto 4$	$401 \mapsto 4$	$402 \mapsto 4$

A game for three players

We can read off a definition for the operation $f\colon \mathbb{Z}_5^3 \to \mathbb{Z}_5$ from this table. For example, we take $24 \mapsto 2$ to indicate that

$$f(2,4,4) = f(4,2,4) = f(4,4,2) = f(4,2,2) = f(2,4,2) = f(2,2,4) = 2.$$

$0 \mapsto 0$	$01 \mapsto 1$	$02 \mapsto 0$	$012 \mapsto 0$	$013 \mapsto 0$
$1 \mapsto 1$	$12 \mapsto 2$	$13 \mapsto 1$	$123 \mapsto 1$	$124 \mapsto 1$
$2 \mapsto 2$	$23 \mapsto 3$	$24 \mapsto 2$	$234 \mapsto 2$	$230 \mapsto 2$
$3 \mapsto 3$	$34 \mapsto 4$	$30 \mapsto 3$	$340 \mapsto 3$	$341 \mapsto 3$
$4 \mapsto 4$	$40 \mapsto 0$	$41 \mapsto 4$	$401 \mapsto 4$	$402 \mapsto 4$

A game for three players

The Cayley table for the 3-magma $\mathbf{A} := (\mathbb{Z}_5, f)$ obtained from this choice of f is given below.

0	0	1	2	3	4
0	0	1	0	3	0
1	1	1	0	0	4
2	0	0	0	2	4
3	3	0	2	3	3
4	0	4	4	3	0

1	0	1	2	3	4
0	1	1	0	0	4
1	1	1	2	1	4
2	0	2	2	1	1
3	0	1	1	1	3
4	4	4	1	3	4

2	0	1	2	3	4
0	0	0	0	2	4
1	0	2	2	1	1
2	0	2	2	3	2
3	2	1	3	3	2
4	4	1	2	2	2

3	0	1	2	3	4
0	3	0	2	3	3
1	0	1	1	1	3
2	2	1	3	3	2
3	3	1	3	3	4
4	3	3	2	4	4

4	0	1	2	3	4
0	0	4	4	3	0
1	4	4	1	3	4
2	4	1	2	2	2
3	3	3	2	4	4
4	0	4	2	4	4

α-action magmas

Definition (α-action magma)

Fix a group $\mathbf{G}$, a set A, and some $n < |A|$. Given a regular group action $\alpha\colon \mathbf{G} \to \mathbf{Perm}(A)$ such that each of the k-extensions of α is free for $1 \leq k \leq n$ let $\Psi_k := \left\{ \operatorname{Orb}(U) \;\middle|\; U \in \binom{A}{k} \right\}$ where $\operatorname{Orb}(U)$ is the orbit of U under α_k. Let $\beta := \{\beta_k\}_{1 \leq k \leq n}$ be a sequence of choice functions $\beta_k\colon \Psi_k \to \binom{A}{k}$ such that $\beta_k(\psi) \in \psi$ for each $\psi \in \Psi_k$. Let $\gamma := \{\gamma_k\}_{1 \leq k \leq n}$ be a sequence of functions $\gamma_k\colon \Psi_k \to A$ such that $\gamma_k(\psi) \in \beta_k(\psi)$ for each $\psi \in \Psi_k$. Let $g\colon \operatorname{Sb}_{\leq n}(A) \to A$ be given by $g(U) := (\alpha(s))(\gamma_k(\psi))$ when $U = (\alpha_k(s))(\beta_k(\psi))$. Define $f\colon A^n \to A$ by $f(a_1, \ldots, a_n) := g(\{a_1, \ldots, a_n\})$. The *$\alpha$-action magma* induced by (β, γ) is $\mathbf{A} := (A, f)$.

α-action magmas are RPS magmas

Theorem

Let $\mathbf{A}$ *be an* α*-action magma induced by* (β, γ)*. We have that* $\mathbf{A} \in \mathrm{RPS}$*.*

Definition (Regular RPS magma)

Let $\mathbf{G}$ be a nontrivial finite group and fix $n < \varpi(|G|)$. We denote by $\mathbf{G}_n(\beta, \gamma)$ the left-multiplication-action n-magma induced by (β, γ), which we refer to as a *regular* RPS *magma*.

Hypergraphs

Definition (Pointed hypergraph)

A *pointed hypergraph* $\mathbf{S} := (S, \sigma, g)$ consists of a hypergraph (S, σ) and a map $g\colon \sigma \to S$ such that for each edge $e \in \sigma$ we have that $g(e) \in e$. The map g is called a *pointing* of (S, σ).

Definition (n-complete hypergraph)

Given a set S we denote by $\mathbf{S}_n$ the *n-complete hypergraph* whose vertex set is S and whose edge set is $\bigcup_{k=1}^{n} \binom{S}{k}$.

Hypertournaments

Definition (Hypertournament)

An *n-hypertournament* is a pointed hypergraph $\mathbf{T} := (T, \tau, g)$ where $(T, \tau) = \mathbf{S}_n$ for some set S.

U	0	1	2	01	12	23	34	40	02	13	24	30	41
$g(U)$	0	1	2	1	2	3	4	0	0	1	2	3	4

U	012	123	234	340	401	013	124	230	341	402
$g(U)$	0	1	2	3	4	0	1	2	3	4

RPS$(5, 3)$ example

Hypertournament magmas

Definition (Hypertournament magma)

Given an n-hypertournament $\mathbf{T} := (T, \tau, g)$ the *hypertournament magma* obtained from $\mathbf{T}$ is the n-magma $\mathbf{A} := (T, f)$ where for $u_1, \ldots, u_n \in T$ we define

$$f(u_1, \ldots, u_n) := g(\{u_1, \ldots, u_n\}).$$

Definition (Hypertournament magma)

A *hypertournament magma* is an n-magma which is conservative and essentially polyadic.

Tournaments

- Tournaments are the $n = 2$ case of a hypertournament.
- Hedrlín and Chvátal introduced the $n = 2$ case of a hypertournament magma in 1965.
- There has been a lot of work on varieties generated by tournament magmas. See for example the survey by Crvenković et al. (1999).

Class containment relationships

Proposition

Let $n > 1$. We have that $\mathrm{RPS}_n \subsetneq \mathrm{PRPS}_n$, $\mathrm{RPS}_n \subsetneq \mathrm{Tour}_n$, *and neither of* PRPS_n *and* Tour_n *contains the other. Moreover,* $\mathrm{RPS}_n = \mathrm{PRPS}_n \cap \mathrm{Tour}_n$.

A generation result

- We denote by $\mathcal{T}_n$ the variety of algebras generated by Tour_n.
- This is equivalent to having

$$\mathcal{T}_n = \mathbf{HSP}(\mathrm{Tour}_n) = \mathrm{Mod}(\mathrm{Id}(\mathrm{Tour}_n)).$$

- Similarly, we define $\mathcal{R}_n$ to be the variety of algebras generated by RPS_n.

A generation result

Theorem

Let $n > 1$. We have that $\mathcal{T}_n = \mathcal{R}_n$. Moreover $\mathcal{T}_n$ is generated by the class of finite regular RPS_n *magmas.*

Proof.

Every finite hypertournament can be embedded in a finite regular balanced hypertournament. □

A generation result

- Trivially we have that $\mathcal{R}_n \leq \mathcal{T}_n$.
- Since n-hypertournament magmas are conservative we have that $\text{Tour}_n \models \epsilon$ if and only if each n-hypertournament magma of order m models epsilon, where m is the number of variables appearing in ϵ.
- It then suffices to show that each finite n-hypertournament magma belongs to $\mathcal{R}_n$.
- It would be very convenient if each finite n-hypertournament embedded into the hypertournament associated to a finite regular RPS magma.
- This turns out to be the case.

A generation result

- Note that in a regular binary RPS magma $\mathbf{G}_2(\beta, \gamma)$ we have that
$$f(e, x) = xf(x^{-1}, e)$$
so exactly one of $f(e, x) = e$ or $f(x^{-1}, e) = e$ holds.
- Note also that the orbit of $\{x, y\}$ contains $\{e, x^{-1}y\}$ and $y^{-1}x, e$, where $x^{-1}y$ and $y^{-1}x$ are inverses.
- We need then only define a map λ specifying for each pair of inverses $\{x, x^{-1}\}$ whether $f(e, x) = e$ or $f(e, x^{-1}) = e$ in order to specify $\mathbf{G}_2(\beta, \gamma)$.
- We can think of $\lambda(\{x, x^{-1}\})$ as choosing the «positive direction» with respect to x and x^{-1}.

A generation result

In order to do this in general we need an n-ary analogue of inverses.

Definition (Obverse k-set)

Given $n > 1$, a nontrivial finite group **G** with $n < \varpi(|G|)$, $1 \leq k \leq n-1$, and $U, V \in \binom{G \setminus \{e\}}{k}$ we say that V is an *obverse* of U when $U = \{a_1, \ldots, a_k\}$ and there exists some $a_i \in U$ such that $V = \{a_i^{-1}\} \cup \{\, a_i^{-1} a_j \mid i \neq j\,\}$. We denote by $\mathrm{Obv}(U)$ the set consisting of all obverses V of U, as well as U itself.

The obverses of a set U are the nonidentity elements in the members of $\mathrm{Orb}(U \cup \{e\}) \setminus (U \cup \{e\})$ which contain e.

A generation result

In order to specify $\mathbf{G}_n(\beta, \gamma)$ it suffices to choose the member $\{a_1, \ldots, a_k\}$ of each collection of obverses for which $f(e, \ldots, e, a_1, \ldots, a_k) = e$.

Definition (*n*-sign function)

Given $n > 1$ and a nontrivial group $\mathbf{G}$ with $n < \varpi(|G|)$ let $\mathrm{Sgn}_n(\mathbf{G})$ denote the set of all choice functions on

$$\left\{ \mathrm{Obv}(U) \;\middle|\; (\exists k \in \{1, \ldots, n-1\}) \left(U \in \binom{G \setminus \{e\}}{k} \right) \right\}.$$

We refer to a member $\lambda \in \mathrm{Sgn}_n(\mathbf{G})$ as an *n-sign function* on $\mathbf{G}$.

We then write $\mathbf{G}_n(\lambda)$ instead of $\mathbf{G}_n(\beta, \gamma)$.

A generation result

- Now we can give the embedding which finishes our proof that $\mathcal{T}_n = \mathcal{R}_n$.
- Consider a finite hypertournament $\mathbf{T} := (T, \tau, g)$.
- Take $\mathbf{G} := \bigoplus_{u \in T} \mathbb{Z}_{\alpha_u}$ where $n < \varpi(\alpha_u)$ and $\mathbb{Z}_{\alpha_u} = \langle u \rangle$.
- We define an n-sign function $\lambda \in \mathrm{Sgn}_n(\mathbf{G})$.
- When $g(\{u_1, \ldots, u_k\}) = u_1$ we define

$$\lambda(\mathrm{Obv}(\{ u_i - u_1 \mid i \neq 1 \})) := \{ u_i - u_1 \mid i \neq 1 \}.$$

- Any values may be chosen for other orbits.
- The n-hypertournament corresponding to $\mathbf{G}_n(\lambda)$ contains a copy of $\mathbf{T}$.

A generation result

- We have now seen that the finite regular RPS n-magmas generate $\mathcal{T}_n = \mathbf{V}(\mathrm{Tour}_n)$.
- In particular we need only use magmas of the form $\mathbf{G}_n(\lambda)$ where:
  1. $\mathbf{G} = \mathbb{Z}^m_{\kappa(n)}$ where $\kappa(n)$ is the least prime strictly greater than n or
  2. $\mathbf{G} = \mathbb{Z}_{\alpha(m)}$ where $\alpha(m) := \prod_{k=\ell}^{m+\ell-1} p_k$ where p_k is the k^{th} prime and $\kappa(n) = p_\ell$.
- In particular, we have that $\mathcal{T}_2$ is generated by regular RPS magmas of the form $(\mathbb{Z}^m_3)_2(\lambda)$.

Automorphisms

Proposition

Let $\mathbf{A} := \mathbf{G}_n(\lambda)$ *be a regular* RPS *magma. There is a canonical embedding of* $\mathbf{G}$ *into* $\mathbf{Aut}(\mathbf{A})$.

Proof.

By construction. □

Exceptional automorphisms

Proposition

For each arity $n \in \mathbb{N}$ *with* $n \neq 1$ *and each group* $\mathbf{G}$ *of composite order* $m \in \mathbb{N}$ *with* $n < \varpi(m)$ *there exists a regular* $\mathrm{RPS}(m, n)$ *magma* $\mathbf{A} := \mathbf{G}_n(\lambda)$ *such that* $|\mathbf{Aut}(\mathbf{A})| > |\mathbf{G}|$.

Proof.

Count the members of $\mathrm{RPS}(\mathbf{G}, n)$ (there are $\prod_{k=1}^{n} k^{\frac{1}{m}\binom{m}{k}}$) and arrive at a contradiction were there no exceptional automorphisms. □

Exceptional automorphisms

Proposition

For each arity $n \in \mathbb{N}$ *and each odd prime* p *such that* $1 \neq n \leq p - 2$ *there exists a regular* $\mathrm{RPS}(p, n)$ *magma* $\mathbf{A} := (\mathbb{Z}_p)_n(\lambda)$ *such that* $|\mathbf{Aut}(\mathbf{A})| > |\mathbf{G}|$.

Proof.

Multiplication by a primitive root modulo p yields an automorphism for an appropriate choice of λ. □

No exceptional automorphisms

Proposition

For each odd prime p and any $\lambda \in \mathrm{Sgn}_{p-1}(\mathbb{Z}_p)$ we have that $\mathbf{Aut}((\mathbb{Z}_p)_{p-1}(\lambda)) \cong \mathbb{Z}_p$.

Corollary

Given an odd prime p the number of isomorphism classes of magmas of the form $(\mathbb{Z}_p)_{p-1}(\lambda)$ is

$$\prod_{k=1}^{p-1} k^{\frac{1}{p}\binom{p}{k}-1}.$$

For $p = 3$ we have 1, for $p = 5$ we have 6, and for $p = 7$ we have 2073600.

Congruences

Theorem

Let $\theta \in \mathrm{Con}(\mathbf{A})$ for a regular $\mathrm{RPS}(m,n)$ magma $\mathbf{A} := \mathbf{G}_n(\lambda)$. Given any $a \in A$ we have that $a/\theta = aH$ for some subgroup $\mathbf{H} \leq \mathbf{G}$.

- One can show by using 2-divisibility that the principal congruence $\theta := \mathrm{Cg}(\{(e,a)\})$ has only one nontrivial class, which is e/θ. This class contains $\mathrm{Sg}^{\mathbf{G}}(\{a\})$.

Congruences

Theorem

Let $\theta \in \mathrm{Con}(\mathbf{A})$ for a regular $\mathrm{RPS}(m,n)$ magma $\mathbf{A} := \mathbf{G}_n(\lambda)$. Given any $a \in A$ we have that $a/\theta = aH$ for some subgroup $\mathbf{H} \leq \mathbf{G}$.

- Any congruence $\theta \in \mathrm{Con}(\mathbf{A})$ has for e/θ a union of cyclic subgroups of $\mathbf{G}$. Suppose that $a, b \in e/\theta$ and $ab \notin e/\theta$.
- Note that $\theta \geq \mathrm{Cg}(\{(e,a),(e,b^{-1})\})$. Observe that

$$\begin{aligned} \mathrm{Cg}(\{(e,a),(e,b^{-1})\}) &= b^{-1}\,\mathrm{Cg}(\{(b,ba),(b,e)\}) \\ &\geq b^{-1}\,\mathrm{Cg}(\{(e,ba)\}) \\ &\geq b^{-1}\,\mathrm{Cg}(\{(e,baba)\}) \\ &\geq \mathrm{Cg}(\{(b^{-1},aba)\}) \end{aligned}$$

so we have that e/θ contains aba.

Congruences

Theorem

Let $\theta \in \mathrm{Con}(\mathbf{A})$ *for a regular* $\mathrm{RPS}(m,n)$ *magma* $\mathbf{A} := \mathbf{G}_n(\lambda)$*. Given any* $a \in A$ *we have that* $a/\theta = aH$ *for some subgroup* $\mathbf{H} \leq \mathbf{G}$*.*

- We have $\langle a \rangle, \langle b \rangle \subset e/\theta$ and $ab \notin e/\theta$ yet $aba \in e/\theta$.
- Since θ is a congruence either ab dominates everything in e/θ ($f(ab,x) = ab$ for all $x \in e/\theta$, which we write as $ab \to x$) or everything in e/θ dominates ab.
- In the former case, we have $ab \to aba$ so $e \to a$.
- We also have $ab \to e$ so $e \to b^{-1}a^{-1}$.
- This implies that $b^{-1} \to b^{-1}a^{-1}$ and hence $e \to a^{-1}$, which is impossible since $e \to a$.
- The argument in the latter case is identical.
- Thus, e/θ is a subgroup of $\mathbf{G}$.

λ-convex subgroups

Definition (λ-convex subgroup)

Given a group $\mathbf{G}$, an n-sign function $\lambda \in \mathrm{Sgn}_n(\mathbf{G})$, and a subgroup $\mathbf{H} \leq \mathbf{G}$ we say that $\mathbf{H}$ is λ-*convex* when there exists some $a \in G$ such that $a/\theta = aH$ for some $\theta \in \mathrm{Con}(\mathbf{G}_n(\lambda))$.

λ-convex subgroups

Proposition

Let $\mathbf{G}$ *be a finite group of order* m *and let* $n < \varpi(m)$. *Take* $\lambda \in \mathrm{Sgn}_n(\mathbf{G})$ *and* $\mathbf{H} \leq \mathbf{G}$. *The following are equivalent:*

1. *The subgroup* $\mathbf{H}$ *is* λ*-convex.*
2. *There exists a congruence* $\psi \in \mathrm{Con}(\mathbf{G}_n(\lambda))$ *such that* $e/\psi = H$.
3. *Given* $1 \leq k \leq n-1$ *and* $b_1, \ldots, b_k \notin H$ *either* $e \to \{b_1 h_1, \ldots, b_k h_k\}$ *for every choice of* $h_1, \ldots, h_k \in H$ *or* $\{b_1 h_1, \ldots, b_k h_k\} \to e$ *for every choice of* $h_1, \ldots, h_k \in H$.

λ-convex subgroups

Theorem

Suppose that $\mathbf{H}, \mathbf{K} \leq \mathbf{G}$ *are both* λ*-convex. We have that* $\mathbf{H} \leq \mathbf{K}$ *or* $\mathbf{K} \leq \mathbf{H}$.

λ-coset poset

Definition (λ-coset poset)

Given $\lambda \in \mathsf{Sgn}_n(\mathbf{G})$ set

$$P_\lambda := \{\, aH \mid a \in G \text{ and } \mathbf{H} \text{ is } \lambda\text{-convex} \,\}$$

and define the *λ-coset poset* to be $\mathbf{P}_\lambda := (P_\lambda, \subset)$.

Lattices of maximal antichains

- Dilworth showed that the maximal antichains of a finite poset form a distributive lattice.
- Freese (1974) gives a succinct treatment of this.
- Given a finite poset $\mathbf{P} := (P, \leq)$ let $\mathbf{L}(\mathbf{P})$ be the lattice whose elements are maximal antichains in $\mathbf{P}$ where if $U, V \in L(\mathbf{P})$ then we say that $U \leq V$ in $\mathbf{L}(\mathbf{P})$ when for every $u \in U$ there exists some $v \in V$ such that $u \leq v$ in $\mathbf{P}$.

Theorem

We have that $\mathbf{Con}(\mathbf{G}_n(\lambda)) \cong \mathbf{L}(\mathbf{P}_\lambda)$.

The search for a basis

- By the year 2000 Ježek, Marković, Maróti, and McKenzie had shown that $\mathcal{T}_2$ was not finitely based.
- To this author's knowledge no equational base for $\mathcal{T}_2$ has ever been described (aside from trivialities like taking $\mathrm{Id}(\mathrm{Tour}_2)$).
- Recall that an identity ϵ in m variables holds in $\mathcal{T}_2$ if and only if it holds in each tournament magma of order m.
- We can use our generation result to see that $\mathcal{T}_2 \models \epsilon$ if and only if ϵ holds in each regular RPS_2 magma of the form $(\mathbb{Z}_3^m)_2(\lambda)$.
- These magmas are much larger than tournaments of order m, but we may have a better chance of understanding their structure and hence their equational theories.

Thank you.