Multiplayer rock-paper-scissors

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Introduction

- In the summer of 2017 I lived in a cave in Yosemite National Park.
- While there I wanted to explain to my friends that I study abstract algebra.
- I realized that rock-paper-scissors was a particularly simple way to do that.

Introduction

We will view the game of RPS as a magma $\mathbf{A} := (A, f)$. We let $A := \{r, p, s\}$ and define a binary operation $f\colon A^2 \to A$ where $f(x, y)$ is the winning item among $\{x, y\}$.

	r	p	s
r	r	p	r
p	p	p	s
s	r	s	s

Introduction

- I also realized that I wanted to be able to play with many of my friends at the same time.
- Naturally, this led me to study the varieties generated by hypertournament algebras.

Properties of RPS

The game RPS is

1. conservative,
2. essentially polyadic,
3. strongly fair, and
4. nondegenerate.

These are the properties we want for a multiplayer game, as well.

What does a multiplayer game mean?

- Suppose we have an *n-ary magma* $\mathbf{A} := (A, f)$ where $f\colon A^n \to A$.
- The *selection game* for $\mathbf{A}$ has n players, $p_1, p_2, \ldots, p_n$.
- Each player p_i simultaneously chooses an item $a_i \in A$.
- The winners of the game are all players who chose $f(a_1, \ldots, a_n)$.

Properties of RPS: Conservativity

- We say that an operation $f\colon A^n \to A$ is *conservative* when for any $a_1, \ldots, a_n \in A$ we have that $f(a_1, \ldots, a_n) \in \{a_1, \ldots, a_n\}$.
- We say that **A** is conservative when each round has at least one winning player.

Properties of RPS: Essential Polyadicity

- We say that an operation $f\colon A^n \to A$ is *essentially polyadic* when there exists some $g\colon \mathrm{Sb}(A) \to A$ such that for any $a_1, \ldots, a_n \in A$ we have $f(a_1, \ldots, a_n) = g(\{a_1, \ldots, a_n\})$.
- We say that **A** is essentially polyadic when a round's winning item is determined solely by which items were played, not taking into account which player played which item or how many players chose a particular item (as long at it was chosen at least once).

Properties of RPS: Strong Fairness

- Let A_k denote the members of A^n which have k distinct components for some $k \in \mathbb{N}$.
- We say that f is *strongly fair* when for all $a, b \in A$ and all $k \in \mathbb{N}$ we have $\left|f^{-1}(a) \cap A_k\right| = \left|f^{-1}(b) \cap A_k\right|$.
- We say that **A** is strongly fair when each item has the same chance of being the winning item when exactly k distinct items are chosen for any $k \in \mathbb{N}$.

Properties of RPS: Nondegeneracy

- We say that f is *nondegenerate* when $|A| > n$.
- In the case that $|A| \leq n$ we have that all members of $A_{|A|}$ have the same set of components.
- If $\mathbf{A}$ is essentially polyadic with $|A| \leq n$ it is impossible for $\mathbf{A}$ to be strongly fair unless $|A| = 1$.

Variants with More Items

The French version of RPS adds one more item: the well. This game is not strongly fair but is conservative and essentially polyadic.

	r	p	s	w
r	r	p	r	w
p	p	p	s	p
s	r	s	s	w
w	w	p	w	w

Variants with More Items

The recent variant Rock-Paper-Scissors-Spock-Lizard is conservative, essentially polyadic, strongly fair, and nondegenerate.

	r	p	s	v	l
r	r	p	r	v	r
p	p	p	s	p	l
s	r	s	s	v	s
v	v	p	v	v	l
l	r	l	s	l	l

Result for Two-Player Games

The only "valid" RPS variants for two players use an odd number of items.

Proposition

Let $\mathbf{A}$ *be a selection game with* $n = 2$ *which is essentially polyadic, strongly fair, and nondegenerate and let* $m := |A|$. *We have that* $m \neq 1$ *is odd. Conversely, for each odd* $m \neq 1$ *there exists such a selection game.*

Proof.

We need $m \mid \binom{m}{2}$. □

PRPS Magmas

Definition (PRPS magma)

Let $\mathbf{A} := (A, f)$ be an n-ary magma. When $\mathbf{A}$ is essentially polyadic, strongly fair, and nondegenerate we say that $\mathbf{A}$ is a PRPS *magma* (read "*pseudo*-RPS *magma*"). When $\mathbf{A}$ is an n-magma of order $m \in \mathbb{N}$ with these properties we say that $\mathbf{A}$ is a PRPS(m, n) *magma*. We also use PRPS and PRPS(m, n) to indicate the classes of such magmas.

Result for Multiplayer Games

Theorem

Let $\mathbf{A} \in \mathrm{PRPS}(m, n)$ *and let* $\varpi(m)$ *denote the least prime dividing* m*. We have that* $n < \varpi(m)$*. Conversely, for each pair* (m, n) *with* $m \neq 1$ *such that* $n < \varpi(m)$ *there exists such a magma.*

Proof.

We need $m \mid \gcd\left(\left\{\binom{m}{2}, \ldots, \binom{m}{n}\right\}\right)$. □

RPS Magmas

Definition (RPS magma)

Let $\mathbf{A} := (A, f)$ be an n-ary magma. When $\mathbf{A}$ is conservative, essentially polyadic, strongly fair, and nondegenerate we say that $\mathbf{A}$ is an RPS *magma*. When $\mathbf{A}$ is an n-magma of order m with these properties we say that $\mathbf{A}$ is an RPS(m, n) *magma*. We also use RPS and RPS(m, n) to indicate the classes of such magmas.

How do I get more RPS magmas?

- In the space below I will show you how to manufacture more of these magmas by hand.

α-action Magmas

Definition (α-action magma)

Fix a group $\mathbf{G}$, a set A, and some $n < |A|$. Given a regular group action $\alpha\colon \mathbf{G} \to \mathbf{Perm}(A)$ such that each of the k-extensions of α is free for $1 \leq k \leq n$ let $\Psi_k := \left\{ \operatorname{Orb}(U) \,\middle|\, U \in \binom{A}{k} \right\}$ where $\operatorname{Orb}(U)$ is the orbit of U under α_k. Let $\beta := \{\beta_k\}_{1\leq k\leq n}$ be a sequence of choice functions $\beta_k\colon \Psi_k \to \binom{A}{k}$ such that $\beta_k(\psi) \in \psi$ for each $\psi \in \Psi_k$. Let $\gamma := \{\gamma_k\}_{1\leq k\leq n}$ be a sequence of functions $\gamma_k\colon \Psi_k \to A$ such that $\gamma_k(\psi) \in \beta_k(\psi)$ for each $\psi \in \Psi_k$. Let $g\colon \mathrm{Sb}_{\leq n}(A) \to A$ be given by $g(U) := (\alpha(s))(\gamma_k(\psi))$ when $U = (\alpha_k(s))(\beta_k(\psi))$. Define $f\colon A^n \to A$ by $f(a_1, \ldots, a_n) := g(\{a_1, \ldots, a_n\})$. The *$\alpha$-action magma* induced by (β, γ) is $\mathbf{A} := (A, f)$.

α-action Magmas are RPS Magmas

Theorem

Let $\mathbf{A}$ *be an* α*-action magma induced by* (β, γ)*. We have that* $\mathbf{A} \in \mathrm{RPS}$*.*

Definition (Regular RPS magma)

Let $\mathbf{G}$ be a nontrivial finite group and fix $n < \varpi(|G|)$. We denote by $\mathbf{G}_n(\beta, \gamma)$ the L-action n-magma induced by (β, γ), which we refer to as a *regular* RPS *magma*.

A Game for Three Players

0	0	1	2	3	4
0	0	1	0	3	0
1	1	1	0	0	4
2	0	0	0	2	4
3	3	0	2	3	3
4	0	4	4	3	0

1	0	1	2	3	4
0	1	1	0	0	4
1	1	1	2	1	4
2	0	2	2	1	1
3	0	1	1	1	3
4	4	4	1	3	4

2	0	1	2	3	4
0	0	0	0	2	4
1	0	2	2	1	1
2	0	2	2	3	2
3	2	1	3	3	2
4	4	1	2	2	2

3	0	1	2	3	4
0	3	0	2	3	3
1	0	1	1	1	3
2	2	1	3	3	2
3	3	1	3	3	4
4	3	3	2	4	4

4	0	1	2	3	4
0	0	4	4	3	0
1	4	4	1	3	4
2	4	1	2	2	2
3	3	3	2	4	4
4	0	4	2	4	4

Hypergraphs

Definition (Pointed hypergraph)

A *pointed hypergraph* $\mathbf{S} := (S, \sigma, g)$ consists of a hypergraph (S, σ) and a map $g \colon \sigma \to S$ such that for each edge $e \in \sigma$ we have that $g(e) \in e$. The map g is called a *pointing* of (S, σ).

Definition (n-complete hypergraph)

Given a set S we denote by $\mathbf{S}_n$ the *n-complete hypergraph* whose vertex set is S and whose edge set is $\bigcup_{k=1}^{n} \binom{S}{k}$.

Hypertournaments

Definition (Hypertournament)

An *n-hypertournament* is a pointed hypergraph $\mathbf{T} := (T, \tau, g)$ where $(T, \tau) = \mathbf{S}_n$ for some set S.

U	0	1	2	01	12	23	34	40	02	13	24	30	41
$g(U)$	0	1	2	1	2	3	4	0	0	1	2	3	4

U	012	123	234	340	401	013	124	230	341	402
$g(U)$	0	1	2	3	4	0	1	2	3	4

RPS$(5, 3)$ example

Hypertournament Magmas

Definition (Hypertournament magma)

Given an n-hypertournament $\mathbf{T} := (T, \tau, g)$ the *hypertournament magma* obtained from $\mathbf{T}$ is the n-magma $\mathbf{A} := (T, f)$ where for $u_1, \ldots, u_n \in T$ we define

$$f(u_1, \ldots, u_n) := g(\{u_1, \ldots, u_n\}).$$

Definition (Hypertournament magma)

A *hypertournament magma* is an n-magma which is conservative and essentially polyadic.

Tournaments

- Tournaments are the $n = 2$ case of a hypertournament.
- Hedrlín and Chvátal introduced the $n = 2$ case of a hypertournament magma in 1965.
- There has been a lot of work on varieties generated by tournament magmas. See for example the survey by Crvenković et al. (1999).

Class Containment Relationships

Proposition

Let $n > 1$. *We have that* $\mathrm{RPS}_n \subsetneq \mathrm{PRPS}_n$, $\mathrm{RPS}_n \subsetneq \mathrm{Tour}_n$, *and neither of* PRPS_n *and* Tour_n *contains the other. Moreover,* $\mathrm{RPS}_n = \mathrm{PRPS}_n \cap \mathrm{Tour}_n$.

A Generation Result

- We denote by $\mathcal{T}_n$ the variety of algebras generated by Tour_n.
- This is equivalent to having

$$\mathcal{T}_n = \mathbf{HSP}(\text{Tour}_n) = \text{Mod}(\text{Id}(\text{Tour}_n)).$$

- Similarly, we define $\mathcal{R}_n$ to be the variety of algebras generated by RPS_n.

A Generation Result

Theorem

Let $n > 1$. We have that $\mathcal{T}_n = \mathcal{R}_n$. Moreover $\mathcal{T}_n$ is generated by the class of finite regular RPS_n *magmas.*

Proof.

Every finite hypertournament can be embedded in a finite regular balanced hypertournament. □

Thank you.