Invariants of structures

Charlotte Aten

University of Denver

2022 September 23

Introduction

- Background story: Bourbaki's structures
- Thesis results
- Isomorphism invariant polynomials

Background story: Bourbaki's structures

- In writing the textbook series *les Éléments de mathématique*, Bourbaki had sought to lay out in the first text of the series, *Theory of Sets* a systematic description of mathematical structures as they would appear throughout the rest of the series.

Background story: Bourbaki's structures

- Basically, they said that a *structure* was a set, say A, equipped with an indexed family $\{f_i\}_{i \in I}$ of *relations* f_i where each f_i was a subset of a set which could be constructed from A by taking Cartesian products and powersets finitely many times.
- For example, a relation on A might be a subset of

$$A \times \mathrm{Sb}(\mathrm{Sb}(A) \times A^{57}) \times \mathrm{Sb}(\mathrm{Sb}(\mathrm{Sb}(A))).$$

Background story: Bourbaki's structures

- Bourbaki defined what we would now call morphisms of these structures and proved several results about them, all of which we would now consider to belong to category theory.
- Once Eilenberg and Mac Lane had established category theory Grothendieck and then Cartier were asked to produce a category theory component for the *Éléments*, although if either did their contribution never made it into the texts.

Background story: Bourbaki's structures

- Discussions in «La Tribu» during the 1950s seem to indicate that Bourbaki felt much of the *Éléments* would have to be rewritten in order to accommodate the new notions from category theory.
- It appeared to be difficult to synthesize the structural and categorical viewpoints together, so the consensus became that this task was not worth the effort.

Thesis results

- In my thesis I presented one possible categorification of Bourbaki's concept of structure.
- The main result in this case is a generalization of a result of Hilbert on symmetric polynomials to the setting of finite structures.
- This generalization has the perhaps surprising implication that any first-order property of a finite structure $\mathbf{A}$ can be checked by counting the number of small substructures $\mathbf{B} \hookrightarrow \mathbf{A}$, where «small» is a function of the logical complexity of the first-order property.

Thesis results

- As Bourbaki imagined, the setup for this is a little involved and is relegated to an appendix.
- That appendix also contains a Yoneda-style embedding theorem which shows that categories of structures built from a set A may always be viewed as having basic relations of the form A^n as in model theory.

Isomorphism invariant polynomials

- A *finite structure* is a pair $\mathbf{A} := (A, \{f_i\}_{i\in I})$ where A is a finite set and the f_i form an I-indexed sequence of relations $f_i \subset A^{\rho(i)}$ where the function $\rho\colon I \to \mathbb{N}$ is the *signature* of $\mathbf{A}$.
- We denote by $\mathbf{Struct}^{\rho}$ the evident category and by $\mathrm{Struct}^{\rho}_{A}$ the collection of all structures of the same signature on the set A, which we call a *kinship class*.
- The class Struct^{ρ} of all structures with signature ρ is likewise called a *similarity class*.

Isomorphism invariant polynomials

Definition (Substructure)

Given a structure $\mathbf{A}$ of signature ρ we refer to a subobject of $\mathbf{A}$ in $\mathbf{Struct}^{\rho}$ as a *substructure* of $\mathbf{A}$.

Isomorphism invariant polynomials

Definition (Finite signature)

We say that a signature $\rho\colon \mathscr{I} \to \mathbf{Fun}(\mathbf{Set}, \mathbf{Set})$ is *finite* when $\mathscr{I}$ has finitely many objects and finitely many morphisms and for each $N \in \mathrm{Ob}(\mathscr{I})$ and each finite set A we have that $\rho_A(N)$ is finite.

Isomorphism invariant polynomials

Definition (Finite kinship class)

When ρ is a finite signature and A is a finite set we say that Struct_A^ρ is a *finite kinship class*.

Isomorphism invariant polynomials

- Given a set of variables X the symmetric group Σ_X of permutations of X acts on the corresponding polynomial algebra $\mathbf{R}[X]$ for some unital commutative ring $\mathbf{R}$.
- The polynomials invariant under this action are the *symmetric polynomials*, which themselves form an $\mathbf{R}$-algebra.
- A classical result of Hilbert is that certain very simple *elementary symmetric polynomials* generate this algebra of all symmetric polynomials.

Isomorphism invariant polynomials

Definition (Monomial $y_{\mathbf{A}}$)

Given a finite signature ρ on an index category $\mathscr{I}$, a finite set A, and a structure $\mathbf{A} := (A, F) \in \mathrm{Struct}_A^{\rho}$ we define

$$y_{\mathbf{A}} := \prod_{N \in \mathrm{Ob}(\mathscr{I})} \prod_{a \in F(N)} x_{N,a}.$$

Isomorphism invariant polynomials

Definition ((ρ, A) polynomial algebra)

Given a commutative ring $\mathbf{R}$, a finite signature ρ, and a finite set A we define the (ρ, A) *polynomial algebra* over $\mathbf{R}$ to be the subalgebra of $\mathbf{R}[X^\rho_A]$ which is generated by Y^ρ_A. We denote this algebra by $\mathbf{Pol}^\rho_A(\mathbf{R})$ and its universe by $\mathrm{Pol}^\rho_A(\mathbf{R})$.

Isomorphism invariant polynomials

Definition (Action υ)

We define a group action $\upsilon\colon \Sigma_A \to \mathbf{Aut}(\mathbf{R}[X_A^\rho])$ by setting $(\upsilon(\sigma))(x_{N,a}) := x_{N,(\rho_\sigma(N))(a)}$ and extending.

Definition (Symmetric polynomial)

A polynomial $p \in \mathrm{Pol}_A^\rho(\mathbf{R})$ is called *symmetric* when for every $\sigma \in \Sigma_A$ we have that $(\upsilon(\sigma))(p) = p$.

Isomorphism invariant polynomials

Definition (Action ζ)

We define a group action $\zeta\colon \Sigma_A \to \Sigma_{\mathsf{Struct}^\rho_A}$ by

$$(\zeta(\sigma))(A, F) := (A, \rho_\sigma \circ F).$$

Isomorphism invariant polynomials

Definition (Isomorphism classes of structures)

We define

$$\mathsf{IsoStr}^{\rho}_{A} := \left\{ \mathsf{Orb}_{\zeta}(\mathbf{A}) \;\middle|\; \mathbf{A} \in \mathsf{Struct}^{\rho}_{A} \right\}.$$

Definition (Elementary symmetric polynomial)

Given a finite signature ρ, a finite set A, and an isomorphism class $\psi \in \mathsf{IsoStr}^{\rho}_{A}$ we define the *elementary symmetric polynomial* of ψ to be

$$s_{\psi} := \sum_{\mathbf{A} \in \psi} y_{\mathbf{A}}.$$

- The elementary symmetric polynomials are symmetric polynomials.

Isomorphism invariant polynomials

Theorem (A. 2022)

Given a polynomial $f \in \mathrm{SymPol}^{\rho}_{A}(\mathbf{R})$ *of degree* d *there exists a polynomial* $g \in R[Z^{\rho}_{A}]$ *of weight at most* d *such that* $f = g|_{Z^{\rho}_{A} = S^{\rho}_{A}}$.

Isomorphism invariant polynomials

- The proof is inductive and follows a proof of Hilbert's result.
- We first show that monomials factor as

$$\prod_{i=1}^{k} y_{\mathbf{A}_i} = y_{\bigvee_{i=1}^{k} \mathbf{A}_i} \mu$$

  where $\mu \in \mathrm{Pol}_A^\rho$.
- We then induct on the size of the universe A.

Isomorphism invariant polynomials

- Supposing we have the result for a universe $B = \{a_1, \dots, a_{n-1}\}$ and we want to show it for $A = \{a_1, \dots, a_n\}$ we define

$$A_n := \bigcup_{N \in \mathrm{Ob}(\mathscr{I})} \{\, x_{N,a} \mid a \in \rho_A(N) \setminus \mathrm{Im}(\rho_\iota(N)) \,\}$$

  to be the collection of variables in X_A^ρ depending on a_n.

- Since

$$f|_{A_n=0} \in \mathrm{SymPol}_B^\rho(\mathbf{R})$$

  there exists some $g_1 \in R[Z_B^\rho]$ of weight at most d such that $f|_{A_n=0} = g_1|_{Z_B^\rho = S_B^\rho}$.

Isomorphism invariant polynomials

- We conclude by arguing that replacing the monomials appearing in this g_1 with the corresponding monomials over A yields a new polynomial, which we abusively also call g_1, such that
$$f = (g_1 + g_2)|_{Z_A^\rho = S_A^\rho}$$
where the additional term g_2 is also a symmetric polynomial.

Thank you!