A multi-linear geometric estimate

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Introduction

- Erdős and Szemerédi made the following significant conjecture in additive number theory: If A is a finite set of integers with $|A| = n$ then either $A + A$ or $A \cdot A$ must have size at least $C_\epsilon n^{2-\epsilon}$ for any $\epsilon > 0$.
- The sum-project conjecture naturally leads one to the consideration of the size of sets of the form $A \cdot A + A \cdot A$.
- Hart and Iosevich previously showed that if $E \subset \mathbb{F}_q^d$ with $|E| > q^{\frac{d+1}{2}}$ then $\mathbb{F}_q^* \subset \varpi(E^2)$ where ϖ is any non-degenerate bilinear form.
- This estimate can be used to show that if $A \subset \mathbb{F}_q^*$ is sufficiently large then

$$F_q^* \subset dA^2 = A \cdot A + \cdots + A \cdot A.$$

- We generalize this estimate to the case that ϖ is a multi-linear form.

Talk outline

- Preliminaries on forms
- The main result
- Applications when $n = 3$ and $d = 2$
- Why $n = 3$ and $d = 2$?
- Proof sketch

Preliminaries on forms

- Given two $\mathbb{F}_q$-vector spaces V and W let $\boxtimes\colon V \times W \to V \otimes W$ denote the canonical map taking (v, w) to $v \otimes w$.
- Note that when E is a subspace of V both $E^{\boxtimes n}$ and $E^{\otimes n}$ are defined and are in general distinct.
- An n-linear form is a linear transformation $\varpi\colon V^{\otimes n} \to \mathbb{F}$ where V is an $\mathbb{F}$-vector space.
- For example, the usual dot product over $\mathbb{F}_q^d$ is a bilinear $(n = 2)$ form.

Preliminaries on forms

Definition (Space of multi-linear forms)

Given a vector space V over a field $\mathbb{F}$ and some $n \in \mathbb{N}$ we denote by $\mathrm{Form}(V, n)$ the dual $\mathbb{F}$-vector space to $V^{\otimes n}$. That is, $\mathrm{Form}(V, n) := \mathrm{Hom}(V^{\otimes n}, \mathbb{F})$.

Preliminaries on forms

Definition (Level set)

Given an $\mathbb{F}$-vector space V, a form $\varpi \in \mathrm{Form}(V, n)$, $E \subset V$, $t \in \mathbb{F}$ we define the *t-level set* of ϖ (with respect to E) to be

$$L_t := \left\{ (z, w) \in E^{\boxtimes(n-1)} \times E \;\middle|\; \varpi(z, w) = t \right\}$$

and we define $\nu(t) := |L_t|$.

Preliminaries on forms

Definition (Evaluation map)

Given a vector space V, some $n \in \mathbb{N}$, some $k \in [n]$, and subspaces $A \leq V^{\otimes(n-1)}$ and $B \leq V$ the k^{th} *evaluation map* on (A, B) is

$$\mathrm{eval}_{k,A,B} \colon \mathrm{Form}(V, n) \otimes B \to \mathrm{Hom}(A, \mathbb{F})$$

is given by

$$(\mathrm{eval}_{k,A,B}(\varpi \otimes y))(x_1, \ldots, x_{n-1}) := \varpi(x_1, \ldots, x_{k-1}, y, x_{k+1}, \ldots, x_{n-1}).$$

Preliminaries on forms

Definition ((A, B)-non-degenerate form)

Given a form $\varpi \in \mathrm{Form}(V, n)$ and subspaces $A \leq V^{\otimes(n-1)}$ and $B \leq V$ we say that ϖ is (A, B)-*non-degenerate* in the k^{th} coordinate when $\mathrm{Ker}(\mathrm{eval}^{\varpi}_{k,A,B}) = 0$.

Preliminaries on forms

- Let $V = \mathbb{F}_q^d$, $A = V^{\otimes(n-1)}$, $B = V$, $n = 3$, and

$$\varpi(x, y, z) = x_1 y_1 z_1 + x_2 y_2 z_2 + \cdots + x_d y_d z_d.$$

  It is not difficult to see that this form is (A, B)-non-degenerate.
- If we keep B the same, change A to $W^{\otimes(n-1)}$, where

$$W = \left\{ x \in \mathbb{F}_q^d \;\middle|\; x_1 = 0 \right\},$$

  and use the same form as above, we get an (A, B)-degenerate form.

Preliminaries on forms

Definition (Projective index)

Given $E \subset \mathbb{F}_q^d$ we say that E has *projective index* α when

$$\frac{\left|\left\{ (a, w) \in (\mathbb{F}_q^* \setminus \{1\}) \times \mathbb{F}_q^d \;\middle|\; w, aw \in E \right\}\right|}{(q-2)\,|E|} \geq \alpha.$$

The main result

Theorem (A., Iosevich (2021))

Suppose that $\varpi \in \mathrm{Form}(q, d, n)$ *for some* $n \geq 2$, *that* $E \subset \mathbb{F}_q^d$, *and that* E *has projective index* α. *If there exists an r-dimensional subspace* A *of* $(\mathbb{F}_q^d)^{\otimes(n-1)}$ *and a subspace* B *of* $\mathbb{F}_q^d$ *such that*

1. $E^{\boxtimes(n-1)} \subset A$,
2. $E \subset B$
3. ϖ *is* (A, B)*-non-degenerate, and*
4. $|E| > q^{\frac{r+n-1}{n}} \left(1 - \alpha\left(1 - \frac{2}{q}\right)\right)^{\frac{1}{n}}$

then $\mathbb{F}_q^* \subset \varpi(E^n)$. *This bound is sharp.*

Applications when $n = 3$ and $d = 2$

Definition (Omphalos)

We say that a set $E \subset \mathbb{F}_q^2$ is a (q, k, ℓ)-*omphalos* when

$$E = \bigcup_{h \in H} E_h$$

where H is a set of k distinct lines through the origin in $\mathbb{F}_q^2$ and each E_h consists of exactly ℓ nonzero points from h.

Applications when $n = 3$ and $d = 2$

Proposition

Suppose that E is a (q, k, ℓ)-omphalos and that ϖ is a non-degenerate ternary form on $\mathbb{F}_q^2$. If

$$k^3\ell^3 > q^6 - (\ell - 1)q^5$$

then $\varpi(E^3) \supset \mathbb{F}_q^*$.

Applications when $n = 3$ and $d = 2$

- Let Γ be a subgroup of $\mathbb{F}_q^*$ of order $\frac{q-1}{s}$ and take $H \subset \mathbb{F}_q^*$ where $|H| = r$ and if $h_1, h_2 \in H$ with $h_1 \neq h_2$ then $h_1\Gamma \neq h_2\Gamma$. That is, let H consist of representatives of r distinct cosets of Γ.
- Define
$$E := \left\{ x(1, y) \in \mathbb{F}_q^2 \;\middle|\; x, y \in H\Gamma \right\}.$$
- Note that Γ is a (q, k, ℓ)-omphalos where
$$k = \ell = \frac{r(q-1)}{s}.$$

Applications when $n = 3$ and $d = 2$

- Taking E from the previous example we have that if

$$(q-1)^6 r^6 + (q-1)s^5 q^5 r - s^6(q^6 + q^5) > 0$$

and ϖ is a non-degenerate ternary form on $\mathbb{F}_q^2$ then $\mathbb{F}_q^* \subset \varpi(E^3)$.

Applications when $n = 3$ and $d = 2$

- Consider the set $E \subset \mathbb{F}_q^2$ from the previous example where $q = 160001$, $s = 20$, and $r = 16$ and take ϖ to be the ternary dot product.
- Since

$$(q-1)^6 r^6 + (q-1)s^5 q^5 r - s^6(q^6 + q^5) > 0$$

in this case and ϖ is non-degenerate we have that every nonzero member of $\mathbb{F}_q^*$ may be written as

$$\varpi(h_1\gamma_1(1, h_2\gamma_2), h_3\gamma_3(1, h_4\gamma_4), h_5\gamma_5(1, h_6\gamma_6))$$

where the h_i are from a fixed set H consisting of $r = 16$ coset representatives of the subgroup Γ of $\mathbb{F}_q^*$ of order $\frac{q-1}{s} = 8000$ and the γ_i are members of Γ.

Applications when $n = 3$ and $d = 2$

- Each member of $\mathbb{F}_q^*$ is of the form

$$h_1 h_3 h_5 \psi_1 (1 + h_2 h_4 h_6 \psi_2)$$

where ψ_1 and ψ_2 are 20^{th} powers in $\mathbb{F}_q$ and the h_i belong to H.

- Moreover,

$$A \cdot A \cdot A + A \cdot A \cdot A \cdot A \cdot A \cdot A \supset \mathbb{F}_q^*$$

when $A = H\Gamma$.

Why $n = 3$ and $d = 2$?

- When $\dim(\mathrm{Span}(E)) = \ell$ we need

$$\ell^{n-1} - n\ell + n < 2 - \log_q(q - \alpha(q-2)).$$

Proof sketch

Theorem (A., Iosevich (2021))

Suppose that $\varpi \in \mathrm{Form}(q, d, n)$ for some $n \geq 2$, that $E \subset \mathbb{F}_q^d$, and that E has projective index α. If there exists an r-dimensional subspace A of $(\mathbb{F}_q^d)^{\otimes(n-1)}$ and a subspace B of $\mathbb{F}_q^d$ such that

1. $E^{\boxtimes(n-1)} \subset A$,
2. $E \subset B$
3. ϖ *is (A, B)-non-degenerate, and*
4. $|E| > q^{\frac{r+n-1}{n}} \left(1 - \alpha\left(1 - \frac{2}{q}\right)\right)^{\frac{1}{n}}$

then $\mathbb{F}_q^* \subset \varpi(E^n)$. This bound is sharp.

Proof sketch

- Write
$$\nu(t) = \sum_{\substack{z \in E^{\boxtimes(n-1)} \\ w \in E}} q^{-1} \sum_{s \in \mathbb{F}_q} \chi(s(\varpi(z,w) - t)).$$
- Thus,
$$\nu(t) = q^{-1} \left| E^{\boxtimes(n-1)} \right| |E| + R$$
where
$$R := \sum_{\substack{z \in E^{\boxtimes(n-1)} \\ w \in E}} q^{-1} \sum_{s \in \mathbb{F}_q^*} \chi(s(\varpi(z,w) - t)).$$

Proof sketch

- View R as a sum in z and apply Cauchy-Schwarz.
- We find that $R^2 \leq U + V$ where

$$U = \left| E^{\boxtimes(n-1)} \right| q^{r-2} \sum_{\substack{s,s' \in \mathbb{F}_q^* \\ w,w' \in \mathbb{F}_q^d \\ sw = s'w'}} \chi(t(s' - s)) E(w) E(w')$$

and

$$V = \left| E^{\boxtimes(n-1)} \right| q^{-2} \sum_{\substack{s,s' \in \mathbb{F}_q^* \\ w,w' \in E \\ sw \neq s'w'}} \chi(t(s' - s)) \sum_{z \in A} \chi(\varpi(z, sw - s'w'))$$

- The (A, B)-nondegeneracy of ϖ and orthogonality of χ gives $V = 0$.

Proof sketch

- We have $R^2 \leq U = C + D$ where

$$C := \left| E^{\boxtimes(n-1)} \right| q^{r-2} \sum_{\substack{s,s' \in \mathbb{F}_q^* \\ w,w' \in \mathbb{F}_q^d \\ sw = s'w' \\ s \neq s'}} \chi(t(s' - s)) E(w) E(w')$$

and

$$D := \left| E^{\boxtimes(n-1)} \right| q^{r-2} \sum_{\substack{s,s' \in \mathbb{F}_q^* \\ w,w' \in \mathbb{F}_q^d \\ sw = s'w' \\ s = s'}} \chi(t(s' - s)) E(w) E(w').$$

- Without using the projective index α we can just note that $C < 0$, but in general this is not enough.

Proof sketch

- Since

$$C \leq - \left| E^{\boxtimes(n-1)} \right| |E| \, q^{r-1} \alpha \left(1 - \frac{2}{q} \right)$$

and

$$D = \left| E^{\boxtimes(n-1)} \right| |E| \, q^{r-1}$$

we have $\nu(t) > 0$ and the result follows.

References

- Derrick Hart and Alex Iosevich. "Sums and products in finite fields: an integral geometric viewpoint". In: *Radon transforms, geometry, and wavelets*. Vol. 464. Contemp. Math. Providence, RI: Amer. Math. Soc., 2008, pp. 129–135