Finite Generation of Families of Structures Equipped with Compatible Group Actions

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Introduction

- Background story 1: FI-module theory
- Background story 2: Bourbaki's structures
- Synergies and bimodules
- Isomorphism invariant polynomials

Background story 1: FI-module theory

- Our first story is quite recent, taking place mostly during the 2010s.
- For some time there were known examples of phenomena called *representation stability* and *homological stability*.
- In both cases a naturally-constructed sequence of objects were known (either representations or spaces) and while their representations or homology groups continued to grow forever, their descriptions «stabilized» into a recognizable pattern.

Background story 1: FI-module theory

- For example, it had been known for some time that when $n \geq 2$ we have that

$$H^1(\mathrm{Conf}_n(\mathbb{C}); \mathbb{C}) \cong \mathbb{C}^{\binom{n}{2}}.$$

- Since each of these cohomology groups is a Σ_n-module, we can decompose $H^1(\mathrm{Conf}_n(\mathbb{C}); \mathbb{C})$ as a sum of irreducible representations.

Background story 1: FI-module theory

- The significant observation here is that when $n \geq 4$ we have that
$$H^1(\mathrm{Conf}_n(\mathbb{C}); \mathbb{C}) = V(0) \oplus V(1) \oplus V(2)$$
where the $V(k)$ are representations induced from those corresponding to the partitions (0), (1), and (2).

Background story 1: FI-module theory

- In 2013 Church and Farb proved that this stabilization in the names of the irreducible representations comprising $H^i(\mathrm{Conf}_n(\mathbb{C}); \mathbb{C})$ as a Σ_n representation occurs for each i.
- Church, Ellenberg, and Farb continued to develop the relevant theory over the next few years, which is the the theory of *FI-modules*.

Background story 1: FI-module theory

- An *FI-module* is a functor from the category FI of finite sets with injections as morphisms into a category $\mathbf{Mod}(\mathbf{R})$ of modules over a commutative unital ring $\mathbf{R}$.
- In 2015 Church, Ellenberg, and Farb proved a Noetherianess result for FI-modules.

Background story 1: FI-module theory

- This led to the 2019 work of Ramos and White on *FI-graphs*, which are functors from the category FI to the category **Grph** of graphs.
- They showed that for those FI-graphs $G_\bullet$ they identified as *vertex-stable* the function

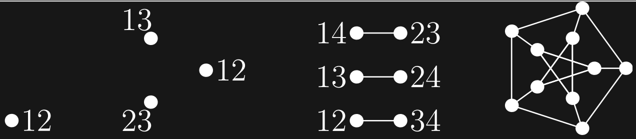
$$n \mapsto \dim_{\mathbb{R}}(H_i(\mathrm{HoCo}(T, G_n); \mathbb{R}))$$

where T is a fixed graph and $\mathrm{HoCo}(T, G_n)$ is the Hom-complex of multi-homomorphisms of T into G_n eventually agrees with a polynomial of degree at most $|V(T)|\, d(i+1)$ where d is the *stable degree* of the vertex-stable FI-graph $G_\bullet$.

Background story 1: FI-module theory

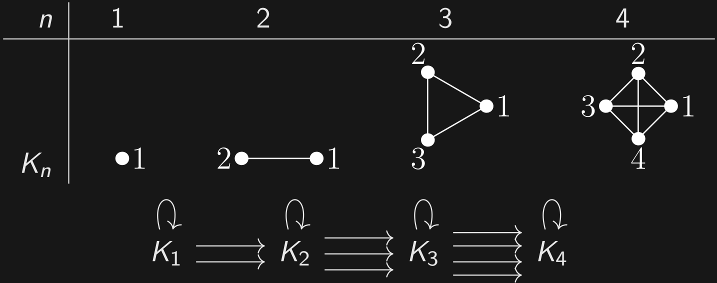
For any fixed r the FI-graph $KG_{\bullet,r}$ is vertex-stable.

n	2	3	4	5
$KG_{n,2}$				



Background story 1: FI-module theory

Any injection from $[m] := \{1, 2, \ldots, m\}$ to $[n] = \{1, 2, \ldots, n\}$ is a homomorphism from K_m to K_n.



Background story 2: Bourbaki's structures

- Our second story takes place during the twentieth century.
- In writing the textbook series *les Éléments de mathématique*, Bourbaki had sought to lay out in the first text of the series, *Theory of Sets* a systematic description of mathematical structures as they would appear throughout the rest of the series.

Background story 2: Bourbaki's structures

- Basically, they said that a *structure* was a set, say A, equipped with an indexed family $\{f_i\}_{i \in I}$ of *relations* f_i where each f_i was a subset of a set which could be constructed from A by taking Cartesian products and powersets finitely many times.
- For example, a relation on A might be a subset of

$$A \times \mathrm{Sb}(\mathrm{Sb}(A) \times A^{57}) \times \mathrm{Sb}(\mathrm{Sb}(\mathrm{Sb}(A))).$$

Background story 2: Bourbaki's structures

- Bourbaki defined what we would now call morphisms of these structures and proved several results about them, all of which we would now consider to belong to category theory.
- Once Eilenberg Mac Lane had established category theory Grothendieck and then Cartier were asked to produce a category theory component for the *Éléments*, although if either did their contribution never made it into the texts.

Background story 2: Bourbaki's structures

- Discussions in «La Tribu» during the 1950s seem to indicate that Bourbaki felt much of the *Éléments* would have to be rewritten in order to accommodate the new notions from category theory.
- It appeared to be difficult to synthesize the structural and categorical viewpoints together, so the consensus became that this task was not worth the effort.

Thesis results

- In my thesis I developed a more general theory which parallels that of FI-modules.
- Instead of a sequence of representations $\{\mathbf{V}_n\}_{n\in\mathbb{N}}$ of the symmetric groups $\{\Sigma_n\}_{n\in\mathbb{N}}$ indexed by the category FI of finite sets with inclusions as morphisms, we consider *synergies*, which are functors from an indexing (or shape) category **S** to the category of groups.
- Building on this, a triad of results about finite generation of corresponding bimodules are proven.

Thesis results

- I also present one possible categorification of Bourbaki's concept of structure here.
- The main result in this case is a generalization of a result of Hilbert on symmetric polynomials to the setting of finite structures.
- This generalization has the perhaps surprising implication that any first-order property of a finite structure $\mathbf{A}$ can be checked by counting the number of embeddings of small substructures $\mathbf{B} \hookrightarrow \mathbf{A}$, where «small» is a function of the logical complexity of the first-order property.

Thesis results

- As Bourbaki imagined, the setup for this is a little involved and is relegated to an appendix.
- That appendix also contains a Yoneda-style embedding theorem which shows that categories of structures built from a set A may always be viewed as having basic relations of the form A^n as in model theory.

Synergies and bimodules

Definition (Synergy)

We refer to a functor $\mathbf{G}\colon \mathbf{S} \to \mathbf{Grp}$ as a *synergy* of shape $\mathbf{S}$ or as an $\mathbf{S}$-*synergy*.

- For $s \in S$ we typically write $\mathbf{G}_s$ rather than $\mathbf{G}(s)$ and given a morphism $f\colon s_1 \to s_2$ in $\mathbf{S}$ we simply write $\check{f}$ rather than $\mathbf{G}(f)$.

Synergies and bimodules

- Many familiar families of groups form synergies.
- The symmetric and alternating groups both form synergies indexed by the natural numbers $\mathbf{N}$.
- The general linear groups $\mathbf{GL}_n(\mathbb{F})$ may be viewed as a synergy indexed by $\mathbf{N}^2$ by taking

$$(\mathbf{GL}(\mathbb{F}))_{i,j} := \mathbf{GL}_{i+j}(\mathbb{F}).$$

Synergies and bimodules

Definition (Unspooling of a synergy)

Given an **S**-synergy **G** the *unspooling* of **G** is the category $\mathcal{G}$ whose objects are the elements of S, whose morphism sets are

$$\mathrm{Hom}_{\mathcal{G}}(s_1, s_2) := \{\, \sigma f \tau \mid \sigma, \tau \in G_{s_2} \text{ and } f \colon s_1 \to s_2 \,\},$$

whose composition map

$$\circ \colon \mathrm{Hom}_{\mathcal{G}}(s_2, s_3) \times \mathrm{Hom}_{\mathcal{G}}(s_1, s_2) \to \mathrm{Hom}_{\mathcal{G}}(s_1, s_3)$$

is given by

$$(\sigma_3 g \tau_3) \circ (\sigma_2 f \tau_2) = \sigma_3 \breve{g}(\sigma_2)(g \circ f)\breve{g}(\tau_2)\tau_3,$$

and whose identity morphisms are those of the form $e\iota e$.

Synergies and bimodules

Definition (Synergy biobject)

Given a synergy $\mathbf{G}$ and a category $\mathscr{C}$ we refer to a functor $\mathbf{V}\colon \mathcal{G} \to \mathscr{C}$ as a $\mathbf{G}$*-biobject in* $\mathscr{C}$.

Definition (Synergy bimodule category)

Given a commutative unital ring $\mathbf{R}$ and a synergy $\mathbf{G}$ we refer to $\mathbf{G}\,\mathbf{Mod}(\mathbf{R})$ as the *category of* $\mathbf{G}$*-bimodules (over* $\mathbf{R}$*)*.

- A symmetric synergy bimodule is an FI-module with a compatible action of the symmetric groups on the right.

Synergies and bimodules

Definition (Regular synergy bimodule)

Given an $\mathbf{S}$-synergy $\mathbf{G}$, a unital commutative ring $\mathbf{R}$, and an $\mathbf{S}$-set Ψ we define the *regular* $\mathbf{G}$*-bimodule*

$$\mathbf{RG}[\Psi] \colon \mathcal{G} \to \mathbf{Mod}(\mathbf{R})$$

by

$$(\mathbf{RG}[\Psi])_s := \mathbf{R}[\{\, \sigma\psi \mid \psi \in \Psi_s \text{ and } \sigma \in G_s \,\}]$$

and

$$\overline{\sigma_2 f \tau_2}(\sigma_1\psi) := \sigma_2 \breve{f}(\sigma_1)\tau_2 \breve{f}(\psi).$$

Synergies and bimodules

Definition (Finitely generated synergy bimodule)

We say that a $\mathbf{G}$-bimodule $\mathbf{V}\colon \mathcal{G} \to \mathbf{Mod}(\mathbf{R})$ is *finitely generated* when there exists an epimorphism $\mathbf{Fr}(\Psi) \twoheadrightarrow \mathbf{V}$ where Ψ is finite.

- A finitely generated synergy bimodule is thus determined by elements lying in a certain collection of modules $\mathbf{V}_s$.

Synergies and bimodules

Definition (Augmentation ideal)

Given a $\mathbf{G}$-bimodule $\mathbf{V}\colon \mathcal{G} \to \mathbf{Mod}(\mathbf{R})$ the *augmentation ideal* $\Theta\mathbf{V}\colon \mathcal{G} \to \mathbf{Mod}(\mathbf{R})$ is the sub-$\mathbf{G}$-bimodule of $\mathbf{V}$ with $(\Theta\mathbf{V})_s$ defined to be the sub-$\mathbf{R}$-module of $\mathbf{V}_s$ generated by

$$\{\, v - \bar{\sigma} v \bar{\tau} \mid v \in V_s,\ \sigma, \tau \in G_s \,\}\,.$$

Synergies and bimodules

Definition (Escalation)

Given a category $\mathbf{S}$ and an endofunctor $\mathring{\xi}\colon \mathbf{S} \to \mathbf{S}$ we refer to a natural transformation $\xi\colon \mathrm{id}_{\mathbf{S}} \to \mathring{\xi}$ as an *escalation* of $\mathbf{S}$.

- Escalations of a poset are isotone maps.
- Escalations of a group are inner automorphisms. (Compare with the work of Cohen et al.)
- The escalations of a category always form a monoid under horizontal composition.

Synergies and bimodules

Definition (Escalation ring)

Given a category $\mathbf{S}$ and a unital commutative ring $\mathbf{R}$ we denote by $\mathbf{R}\,\mathbf{Esc}(\mathbf{S})$ the *escalation ring (of* $\mathbf{S}$ *over* $\mathbf{R}$*)*, which is the monoid ring of $\mathbf{Esc}(\mathbf{S})$ over $\mathbf{R}$.

Definition (Ring of a set of escalations)

Given a category $\mathbf{S}$ and some $\Xi \subset \mathrm{Esc}(\mathbf{S})$ we denote by $\mathbf{R}\{\Xi\}$ the subring of $\mathbf{R}\,\mathbf{Esc}(\mathbf{S})$ generated by $R \cup \Xi$.

Synergies and bimodules

Definition (Coinvariants module)

Let $\mathbf{G}$ be an $\mathbf{S}$-synergy which has a generating set Ξ and let $\mathbf{R}$ be a unital commutative ring. Given a $\mathbf{G}$-bimodule $\mathbf{V}\colon \mathcal{G} \to \mathbf{Mod}(\mathbf{R})$ the Ξ-*coinvariants module* $\Phi\mathbf{V}$ is an S-graded $\mathbf{R}\{\Xi\}$-module whose s^{th} component is

$$(\Phi\mathbf{V})_s := \mathbf{V}_s/(\Theta\mathbf{V})_s$$

and for which $\xi \in \Xi$ acts as a map

$$\dot{\xi}_s\colon (\Phi\mathbf{V})_s \to (\Phi\mathbf{V})_{\mathring{\xi}(s)}$$

which is given by

$$\dot{\xi}_s(v/(\Theta V)_s) := \bar{\xi}_s(v)/(\Theta V)_{\mathring{\xi}(s)}.$$

Synergies and bimodules

Definition (Noetherian category)

Given a category $\mathbf{S}$ which is finitely generated by (Ξ, B) and a unital commutative ring $\mathbf{R}$ we say that $\mathbf{S}$ is $(\mathbf{R}, \Xi)$-*Noetherian* (or *Noetherian (over* $\mathbf{R}$ *with respect to* Ξ*)*) when $\mathbf{R}\{\Xi\}$ is a Noetherian ring.

Synergies and bimodules

Proposition (A. 2022)

If $\mathbf{G}$ *is a synergy then for any finite* $\mathbf{S}$*-set* Ψ *we have that* $\mathbf{RG}[\Psi]$ *is finitely generated. If* $\mathbf{G}$ *is NFG by* (Ω, Ω', B) *and* Ψ *is finite with finite generating set* Ψ' *whose associated base is* B *then* $\Theta\mathbf{G}[\Psi]$ *is finitely generated.*

- We get a relatively explicit bound on the size of a finite generating set for $\Theta\mathbf{G}[\Psi]$ since we have that

$$\left|\Psi^{\Omega}\right| \leq \left|(\Psi')^{\Omega'}\right| \leq 2\sum_{s\in B} \left|\Psi' \cap \Psi_s\right| \left|\Omega' \cap \Omega_s\right|.$$

Synergies and bimodules

Theorem (A. 2022)

Suppose that $\mathbf{G}$ *is an* $\mathbf{S}$*-synergy and that* $\mathbf{V}\colon \mathcal{G} \to \mathbf{Mod}(\mathbf{R})$ *is a* $\mathbf{G}$*-bimodule with* $\mathbf{W} \leq \mathbf{V}$*. If*

1. $\Theta\mathbf{W}$ *is finitely generated with witness* $q_\Theta\colon \mathbf{Fr}(\Psi_\Theta) \twoheadrightarrow \mathbf{V}$ *where* Ψ_Θ *is finite with finite generating set* Ψ'_Θ *whose associated base is* B_Θ*,*
2. $\mathbb{Q} \leq \mathbf{R}$,
3. *all the groups* $\mathbf{G}_s$ *are torsion,*
4. $\mathbf{S}$ *is* $(\mathbf{R}, \Xi)$*-Noetherian,*
5. $\mathbf{V}$ *is finitely generated with witness* $q\colon \mathbf{Fr}(\Psi) \twoheadrightarrow \mathbf{V}$ *where* Ψ *is finite with finite generating set* Ψ' *whose associated base is* B*,*
6. $\mathbf{S}$ *is generated by* (Ξ, B)

then $\mathbf{W}$ *is finitely generated.*

Synergies and bimodules

Theorem (A. 2022)

Sub-Σ-bimodules of $\mathbb{C}\Sigma[1]$ are finitely generated.

Isomorphism invariant polynomials

- A *finite structure* is a pair $\mathbf{A} := (A, \{f_i\}_{i \in I})$ where A is a finite set and the f_i form an I-indexed sequence of relations $f_i \subset A^{\rho(i)}$ where the function $\rho\colon I \to \mathbb{N}$ is the *signature* of $\mathbf{A}$.
- We denote by $\mathbf{Struct}^{\rho}$ the evident category and by $\mathrm{Struct}^{\rho}_{A}$ the collection of all structures of the same signature on the set A, which we call a *kinship class*.
- The class Struct^{ρ} of all structures with signature ρ is likewise called a *similarity class*.

Isomorphism invariant polynomials

Definition (Substructure)

Given a structure $\mathbf{A}$ of signature ρ we refer to a subobject of $\mathbf{A}$ in $\mathbf{Struct}^{\rho}$ as a *substructure* of $\mathbf{A}$.

Isomorphism invariant polynomials

Definition (Finite signature)

We say that a signature $\rho\colon \mathscr{I} \to \mathbf{Fun}(\mathbf{Set}, \mathbf{Set})$ is *finite* when $\mathscr{I}$ has finitely many objects and finitely many morphisms and for each $N \in \mathrm{Ob}(\mathscr{I})$ and each finite set A we have that $\rho_A(N)$ is finite.

Isomorphism invariant polynomials

Definition (Finite kinship class)

When ρ is a finite signature and A is a finite set we say that $\mathrm{Struct}^{\rho}_{A}$ is a *finite kinship class*.

Isomorphism invariant polynomials

- Given a set of variables X the symmetric group Σ_X of permutations of X acts on the corresponding polynomial algebra $\mathbf{R}[X]$ for some unital commutative ring $\mathbf{R}$.
- The polynomials invariant under this action are the *symmetric polynomials*, which themselves form an $\mathbf{R}$-algebra.
- A classical result of Hilbert is that certain very simple *elementary symmetric polynomials* generate this algebra of all symmetric polynomials.

Isomorphism invariant polynomials

Definition (Monomial $y_{\mathbf{A}}$)

Given a finite signature ρ on an index category $\mathscr{I}$, a finite set A, and a structure $\mathbf{A} := (A, F) \in \mathrm{Struct}_A^\rho$ we define

$$y_{\mathbf{A}} := \prod_{N \in \mathrm{Ob}(\mathscr{I})} \prod_{a \in F(N)} x_{N,a}.$$

Isomorphism invariant polynomials

Definition ((ρ, A) polynomial algebra)

Given a commutative ring $\mathbf{R}$, a finite signature ρ, and a finite set A we define the (ρ, A) *polynomial algebra* over $\mathbf{R}$ to be the subalgebra of $\mathbf{R}[X_A^\rho]$ which is generated by Y_A^ρ. We denote this algebra by $\mathbf{Pol}_A^\rho(\mathbf{R})$ and its universe by $\mathrm{Pol}_A^\rho(\mathbf{R})$.

Isomorphism invariant polynomials

Definition (Action υ)

We define a group action $\upsilon \colon \Sigma_A \to \mathbf{Aut}(\mathbf{R}[X_A^\rho])$ by setting $(\upsilon(\sigma))(x_{N,a}) := x_{N,(\rho_\sigma(N))(a)}$ and extending.

Definition (Symmetric polynomial)

A polynomial $p \in \mathrm{Pol}_A^\rho(\mathbf{R})$ is called *symmetric* when for every $\sigma \in \Sigma_A$ we have that $(\upsilon(\sigma))(p) = p$.

Isomorphism invariant polynomials

Definition (Action ζ)

We define a group action $\zeta\colon \Sigma_A \to \Sigma_{\mathrm{Struct}^{\rho}_{A}}$ by

$$(\zeta(\sigma))(A, F) := (A, \rho_\sigma \circ F).$$

Isomorphism invariant polynomials

Definition (Isomorphism classes of structures)

We define

$$\mathsf{IsoStr}_A^\rho := \{\, \mathsf{Orb}_\zeta(\mathbf{A}) \mid \mathbf{A} \in \mathsf{Struct}_A^\rho \,\}.$$

Definition (Elementary symmetric polynomial)

Given a finite signature ρ, a finite set A, and an isomorphism class $\psi \in \mathsf{IsoStr}_A^\rho$ we define the *elementary symmetric polynomial* of ψ to be

$$s_\psi := \sum_{\mathbf{A} \in \psi} y_{\mathbf{A}}.$$

- The elementary symmetric polynomials are symmetric polynomials.

Isomorphism invariant polynomials

Theorem (A. 2022)

Given a polynomial $f \in \mathrm{SymPol}^{\rho}_{A}(\mathbf{R})$ *of degree* d *there exists a polynomial* $g \in R[Z^{\rho}_{A}]$ *of weight at most* d *such that* $f = g|_{Z^{\rho}_{A} = S^{\rho}_{A}}$.

Thank you!