On the construction of manifolds from n-ary quasigroups

Charlotte Aten (joint work with Semin Yoo)

University of Denver

2023 March 22

Introduction

- In the 2010s Herman and Pakianathan introduced a functorial construction of closed surfaces from noncommutative finite groups.
- Semin Yoo and I decided to produce an n-dimensional generalization.
- The two main challenges in doing this were finding an appropriate analogue of noncommutative groups and in desingularizing the n-dimensional pseudomanifolds which arose at the first stage of our construction.
- Ultimately we found that every orientable triangulable manifold could be manufactured in the manner we described.

Introduction

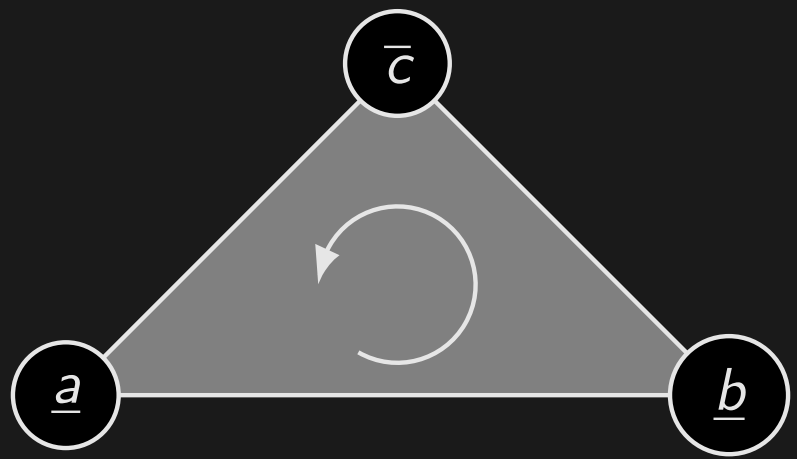
- Our preprint “Orientable smooth manifolds are essentially quasigroups” may be found at https://arxiv.org/abs/2110.05660.
- Relevant code appears at https://github.com/caten2/SimplexBuilder.

Talk outline

- Herman and Pakianathan's construction
- Quasigroups instead of groups
- The n-ary case
- The first functor: Open serenation
- The second functor: Serenation
- The Evans Conjecture and Latin cubes

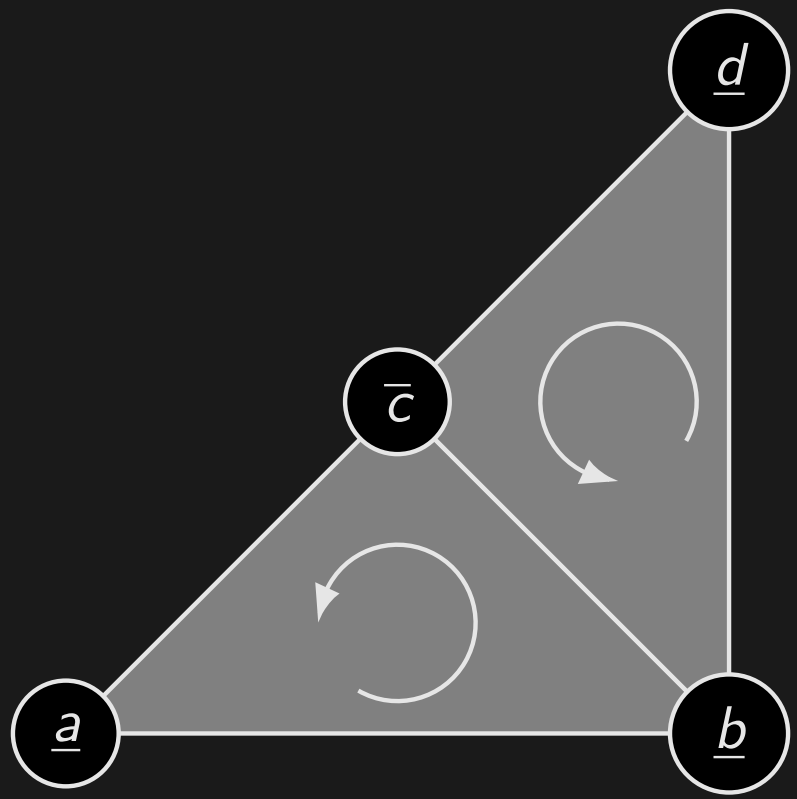
Herman and Pakianathan's construction

- Consider a set A equipped with a binary operation $f\colon A^2 \to A$.
- Given elements $a, b \in A$ we can represent that $f(a, b) = c$ with a corresponding triangle.



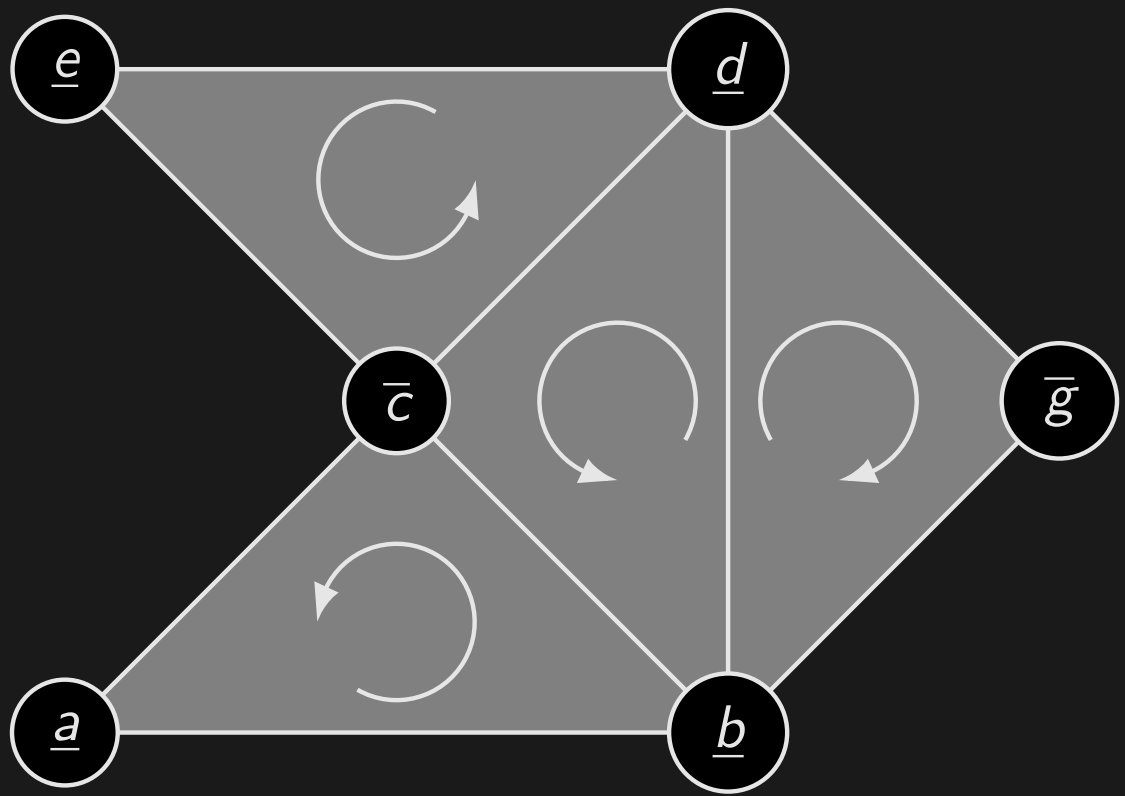
Herman and Pakianathan's construction

- If it also happens that $d \in A$ with $f(b, d) = c$ then we can continue our picture by adding another triangle.



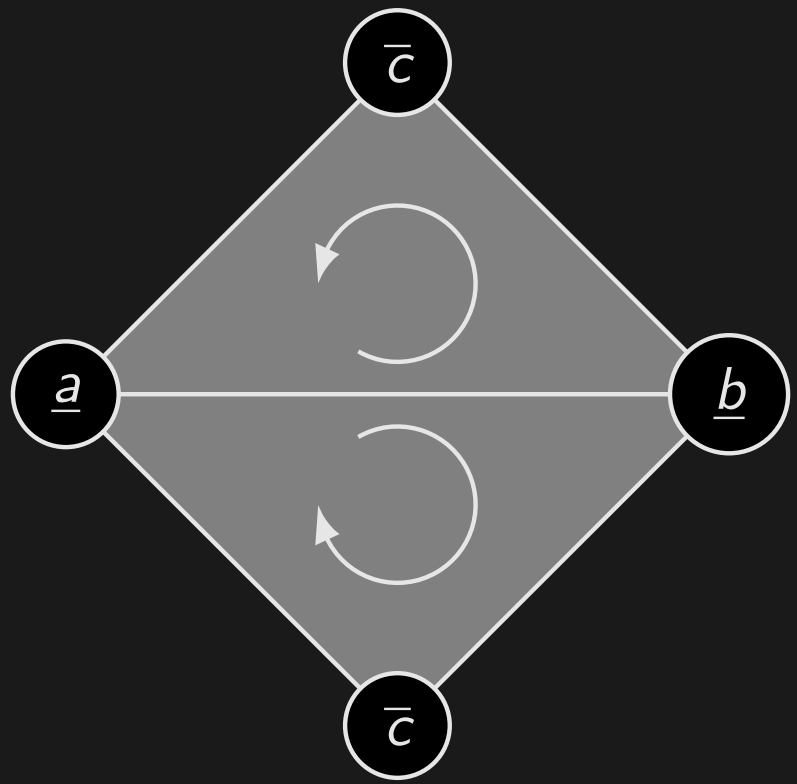
Herman and Pakianathan's construction

- We may continue in this fashion, building a simplicial complex whose vertices are $\underline{x}$ and $\overline{x}$ for $x \in A$ and whose facets are of the form $\left\{\underline{x}, \underline{y}, \overline{f(x,y)}\right\}$.



Herman and Pakianathan's construction

- If it happens that $f(a,b) = f(b,a)$ then we will have «two» faces with the same vertices.
- Solution: Only form facets $\left\{\underline{a}, \underline{b}, \overline{f(a,b)}\right\}$ when a and b do not commute under f.



Herman and Pakianathan's construction

- Consider the quaternion group $\mathbf{G}$ of order 8 whose universe is $G := \{\pm 1, \pm i, \pm j, \pm k\}$.
- We begin by picking out all the pairs of elements $(x, y) \in G^2$ so that $xy \neq yx$. We call this collection $\mathrm{NCT}(\mathbf{G})$.
- We define $\mathrm{In}(\mathbf{G})$ to be all the elements of G which are entries in some pair $(x, y) \in \mathrm{NCT}(\mathbf{G})$.
- Similarly, $\mathrm{Out}(\mathbf{G})$ is defined to be all the members of G of the form xy where $(x, y) \in \mathrm{NCT}(\mathbf{G})$.

Herman and Pakianathan's construction

- In this case we have

$$\mathrm{NCT}(\mathbf{G}) = \left\{ (\pm u, \pm v) \,\middle|\, \{u, v\} \in \binom{\{i, j, k\}}{2} \right\}$$

so

$$\mathrm{In}(\mathbf{G}) = \{\pm i, \pm j, \pm k\}$$

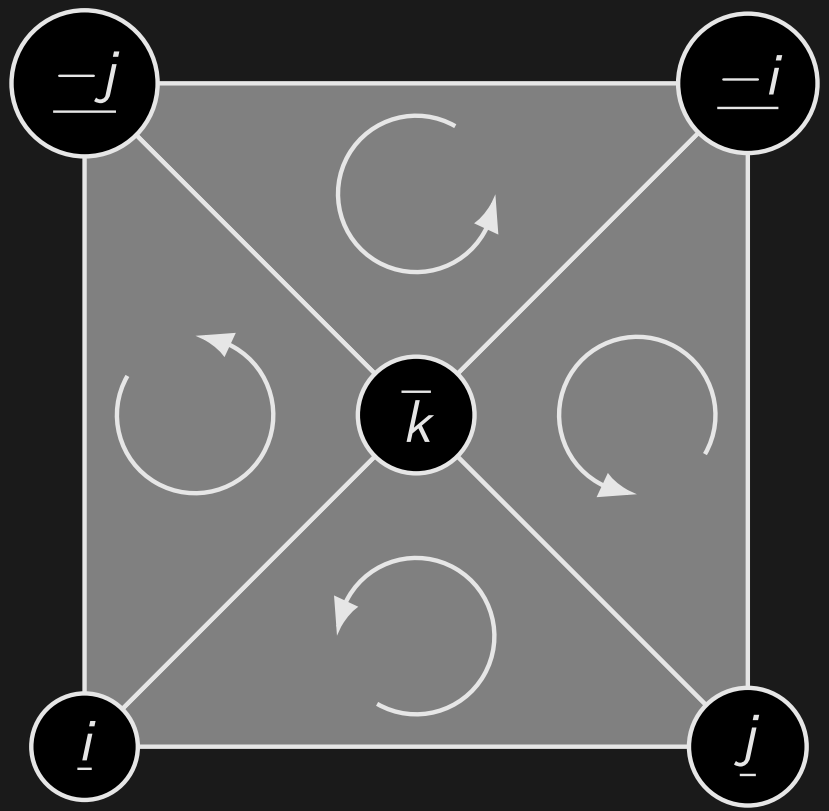
and

$$\mathrm{Out}(\mathbf{G}) = \{\pm i, \pm j, \pm k\}\,.$$

- From this data we form a simplicial complex (actually a 2-pseudomanifold) whose facets are of the form $\left\{\underline{x}, \underline{y}, \overline{xy}\right\}$ where $(x, y) \in \mathrm{NCT}(\mathbf{G})$.

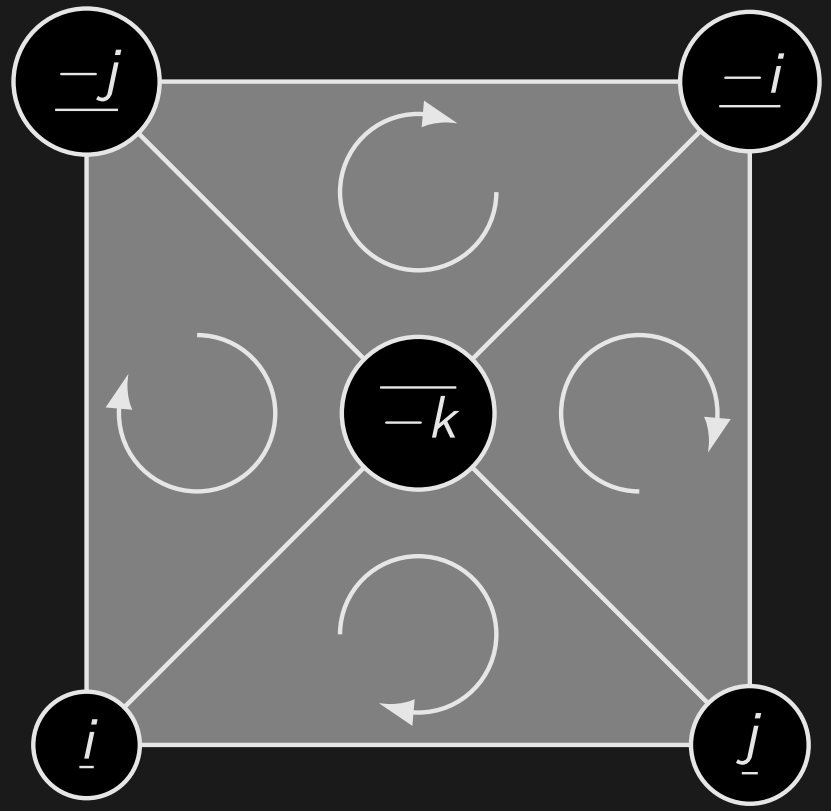
Herman and Pakianathan's construction

- One «sheet» of this complex is pictured below.



Herman and Pakianathan's construction

- There is a partner sheet carrying the opposite orientation on the cycle formed by the input vertices.

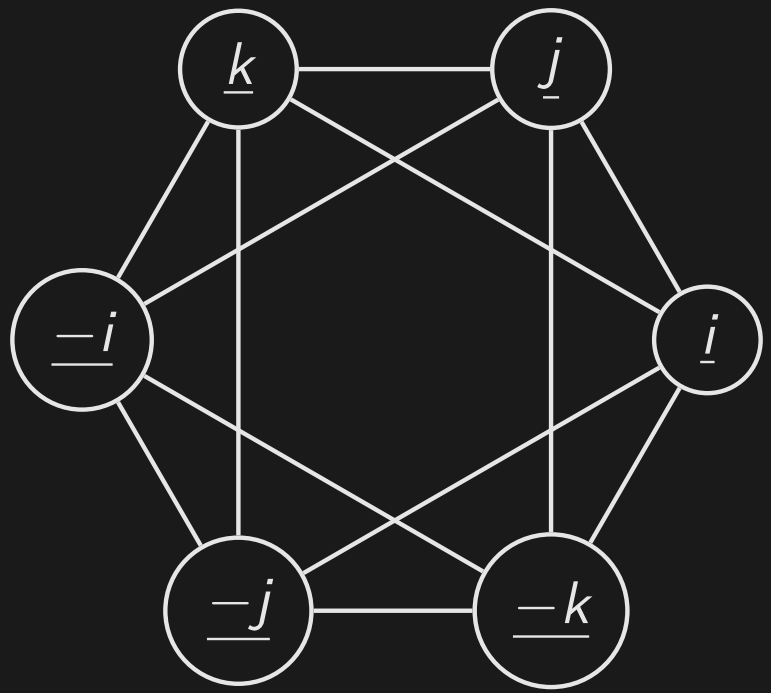


Herman and Pakianathan's construction

- The three 4-cycles

$$(\underline{i}, \underline{j}, \underline{-i}, \underline{-j}),\ (\underline{i}, \underline{k}, \underline{-i}, \underline{-k}),\ \text{and}\ (\underline{j}, \underline{k}, \underline{-j}, \underline{-k}).$$

  each carry an octahedron.



Herman and Pakianathan's construction

- This simplicial complex, which we call **Sim**(**G**) and Herman and Pakianathan called $X(Q_8)$, consists of three 2-spheres, each pair of which is glued at two points.
- Deleting these points to disjointize the spheres and filling the resulting holes yields the manifold we call **Ser**(**G**) and Herman and Pakianathan called $Y(Q_8)$.
- In this case **Ser**(**G**) is the disjoint union of three 2-spheres.

Quasigroups instead of groups

Definition (Quasigroup)

A *(binary) quasigroup* is a magma $\mathbf{A} \coloneqq (A, f\colon A^2 \to A)$ such that if any two of the variables x, y, and z are fixed the equation

$$f(x, y) = z$$

has a unique solution.

- That is, a quasigroup is a magma whose Cayley table is a Latin square, where each entry occurs once in each row and each column.
- All groups are quasigroups, but quasigroups need not have identities or be associative.

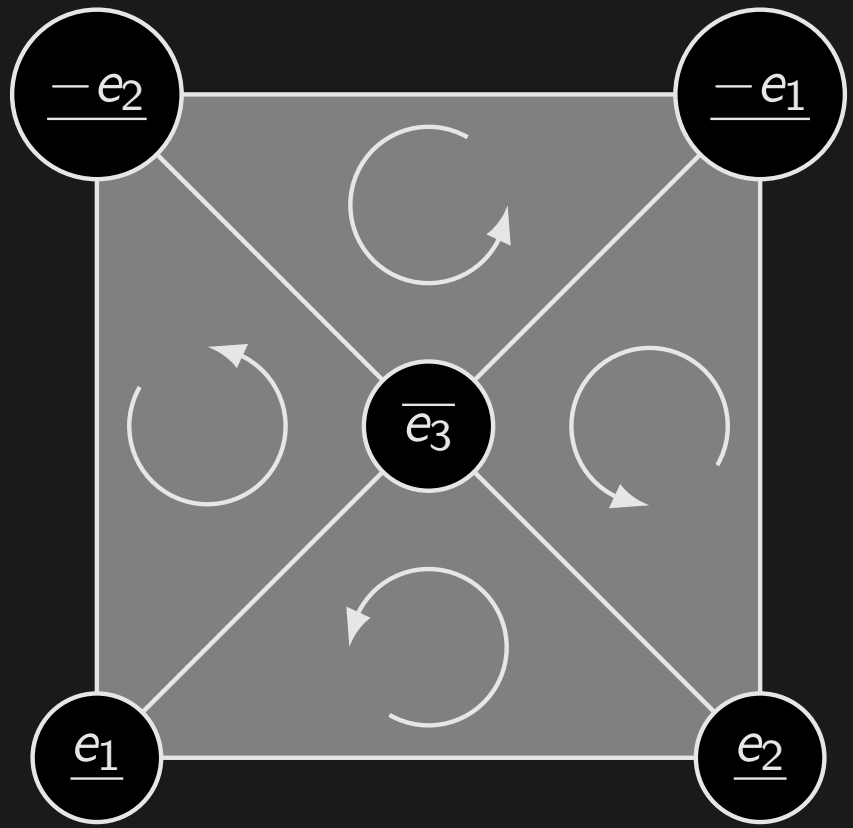
Quasigroups instead of groups

- We didn't need the fact that the quaternion group was associative (or had an identity element) in order to perform this construction.
- Consider now the octonion loop $\mathbf{L}$ of order 16 whose universe is $L := \{\pm e_0, \pm e_1, \ldots, \pm e_7\}$.
- In this case

$$\mathrm{NCT}(\mathbf{L}) = \{\, (\pm e_i, \pm e_j) \mid i \neq j \text{ and } i, j \neq 0 \,\}.$$

Quasigroups instead of groups

- We can again form sheets as we did for the quaternion group **G** previously.



Quasigroups instead of groups

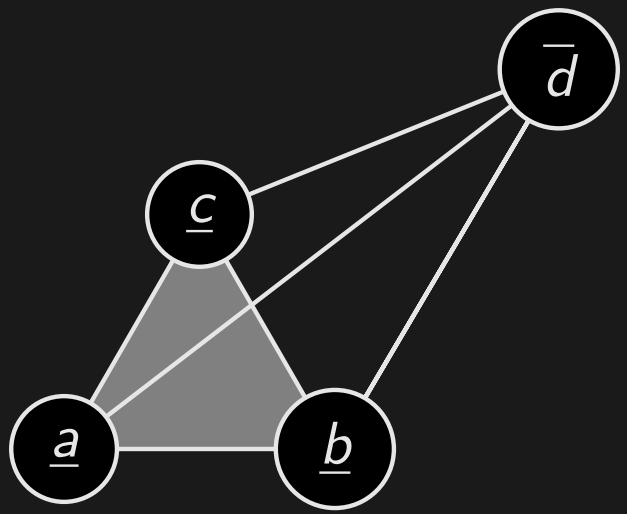
- These sheets pair up to form octahedra as before.
- We find that **Sim**(**L**) consists of twenty-one 2-spheres which are glued together along their vertices in some manner.
- If we disjointize by deleting vertices and then fill in the resulting holes we obtain the manifold **Ser**(**L**), which is the disjoint union of twenty-one 2-spheres.

Quasigroups instead of groups

- It is an immediate corollary of the Evans Conjecture that every compact orientable surface is a component of $\mathbf{Ser}(\mathbf{Q})$ for some finite quasigroup $\mathbf{Q}$.
- We'll come back to this later.

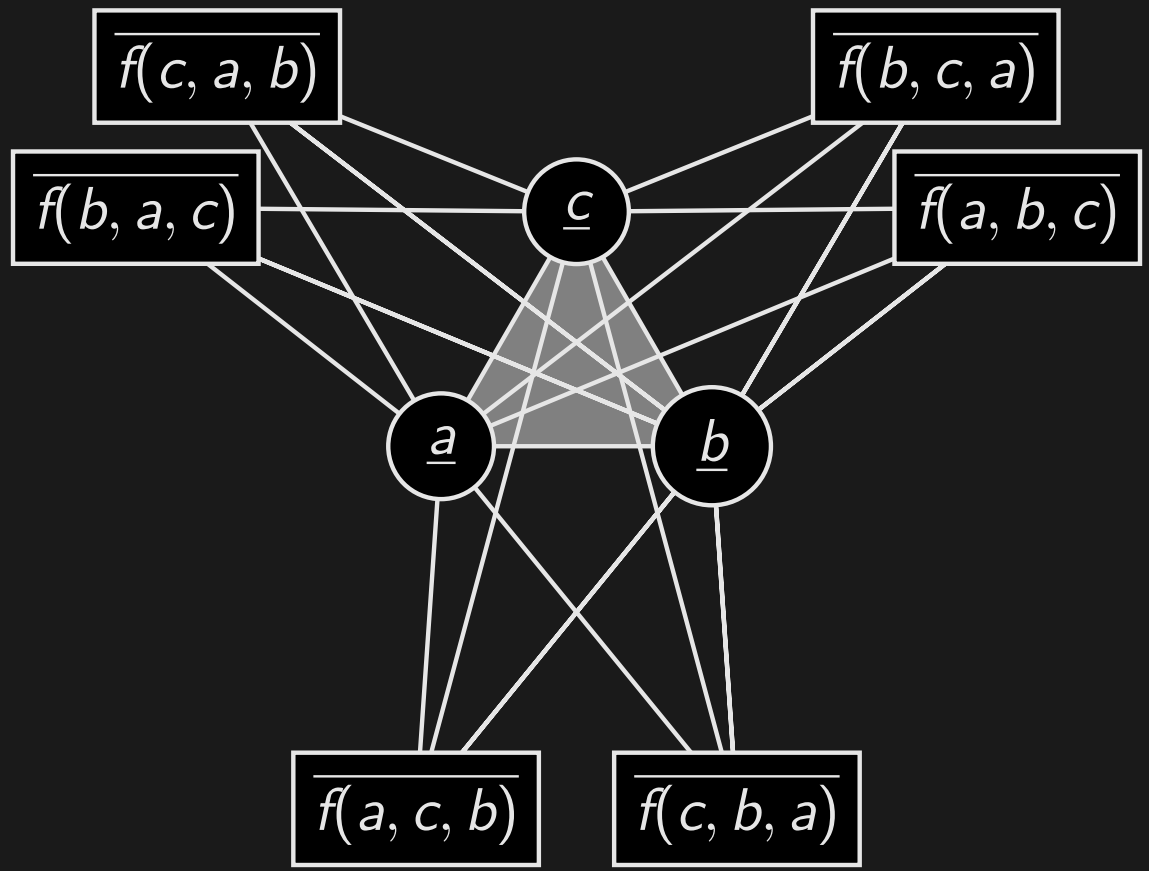
The n-ary case

- We can generalize this situation to the creation of a n-dimensional pseudomanifold from an n-ary operation $f\colon A^n \to A$.
- The case $n = 3$ is illustrative.
- Given elements $a, b, c, d \in A$ we can represent that $f(a, b, c) = d$ with a corresponding tetrahedron.



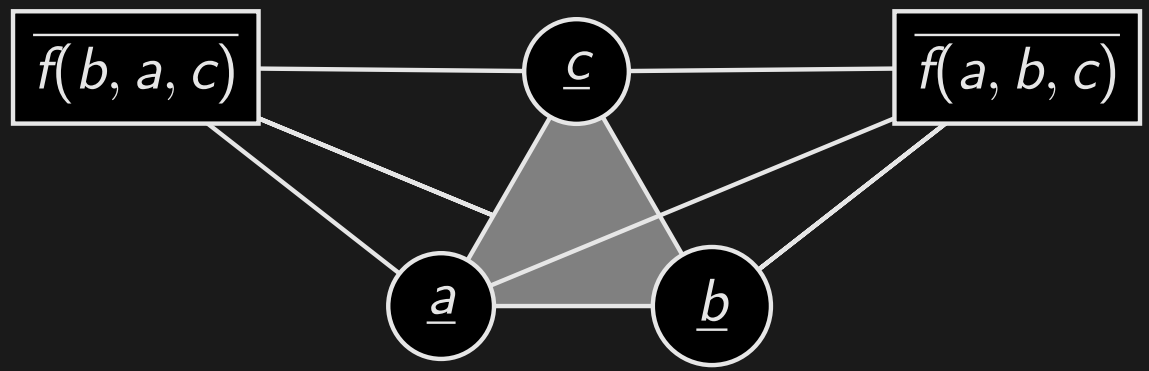
The n-ary case

- We now have a different problem: Up to six tetrahedra could meet at the triangle $\{\underline{a}, \underline{b}, \underline{c}\}$.



The n-ary case

- Solution: Require that f is invariant under even permutations of its arguments.
- In this case, $f(a, b, c) = f(b, c, a) = f(c, a, b)$ but in general $f(a, b, c) \neq f(b, a, c)$.



The n-ary case

Definition (Quasigroup)

An *n-quasigroup* is an n-magma $\mathbf{A} := (A, f\colon A^n \to A)$ such that if any $n-1$ of the variables $x_1, \ldots, x_n, y$ are fixed the equation

$$f(x_1, \ldots, x_n) = y$$

has a unique solution.

- That is, an n-quasigroup is an n-magma whose Cayley table is a Latin n-cube.
- All n-ary groups are quasigroups, but quasigroups need not be associative.

The n-ary case

- Given any group $\mathbf{G}$ the n-ary multiplication

$$f(x_1, \ldots, x_n) := x_1 \cdots x_n$$

  is a quasigroup operation on G.

The n-ary case

- We say that an n-quasigroup $\mathbf{A}$ is *commutative* when for all $x_1, \ldots, x_n \in A$ and all $\sigma \in \mathrm{Perm}_n$ we have

$$f(x_1, \ldots, x_n) = f(x_{\sigma(1)}, \ldots, x_{\sigma(n)}).$$

- We say that an n-quasigroup $\mathbf{A}$ is *alternating* when for all $x_1, \ldots, x_n \in A$ and all $\sigma \in \mathrm{Alt}_n$ we have

$$f(x_1, \ldots, x_n) = f(x_{\sigma(1)}, \ldots, x_{\sigma(n)}).$$

- Our "correct" analogue of the variety of groups will be the variety AQ_n of alternating n-ary quasigroups.

The n-ary case

- There are nontrivial members of AQ_n for each n, but the easiest examples are either commutative (take the n-ary multiplication for an abelian group) or infinite (the free alternating quasigroups).
- For $n \geq 3$, every alternating n-ary group is commutative.
- We tediously found the following example by hand:

The n-ary case

- Take $S := (\mathbb{Z}/5\mathbb{Z})^3$ and define $h\colon \mathbb{Z}/5\mathbb{Z} \times \mathbf{Alt}_3 \to \mathbf{Perm}_S$ by

$$(h(k,\sigma))(x_1,x_2,x_3) := (x_{\sigma(1)}+k, x_{\sigma(2)}+k, x_{\sigma(3)}+k).$$

  There are 7 members of $\mathrm{Orb}(h)$. One system of orbit representatives is:

$$\{000, 011, 022, 012, 021, 013, 031\}.$$

The n-ary case

- Let $A := \mathbb{Z}/5\mathbb{Z}$ and define a ternary operation $f\colon A^3 \to A$ so that

$$f((h(k,\sigma))(x_1,x_2,x_3)) = f(x_1,x_2,x_3) + k$$

and f is defined on the above set of orbit representatives as follows.

xyz	$f(x,y,z)$
000	0
011	0
022	0
012	3
021	4
013	4
031	2

The n-ary case

- We reached out to Jonathan Smith to see if anyone had studied the varieties of alternating n-quasigroups before, but it seemed that no one had.
- He did, however, give us an example which we generalized into an *alternating product* construction which takes an n-ary commutative quasigroup and an $(n+1)$-ary commutative quasigroup and yields an n-ary alternating quasigroup which is typically not commutative.

The n-ary case

Definition (Commuting tuple)

Given $\mathbf{A} \coloneqq (A, f) \in \mathrm{AQ}_n$ we say that $a \in A^n$ *commutes* (or is a *commuting tuple*) in $\mathbf{A}$ when we have for each $\sigma \in \mathrm{Perm}_n$ that

$$f(a) = f(a_{\sigma(1)}, \ldots, a_{\sigma(n)}).$$

Definition (Set of noncommuting tuples)

Given $\mathbf{A} \coloneqq (A, f) \in \mathrm{AQ}_n$ we define the *noncommuting tuples* $\mathrm{NCT}(\mathbf{A})$ of $\mathbf{A}$ by

$$\mathrm{NCT}(\mathbf{A}) \coloneqq \{\, a \in A^n \mid a \text{ does not commute in } \mathbf{A} \,\}.$$

The n-ary case

Definition (NC homomorphism)

We say that a homomorphism $h\colon \mathbf{A}_1 \to \mathbf{A}_2$ of alternating quasigroups is an *NC homomorphism* (or a *noncommuting homomorphism*) when for each $a \in \mathrm{NCT}(\mathbf{A}_1)$ we have that

$$h(a) = (h(a_1), \ldots, h(a_n)) \in \mathrm{NCT}(\mathbf{A}_2).$$

- All embeddings are NC homomorphisms, but there are other examples as well.
- The class of n-ary alternating quasigroups equipped with NC homomorphisms forms the category $\mathbf{NCAQ}_n$.

The first functor: Open serenation

- Our first construction gives a functor

$$\mathbf{OSer}_n \colon \mathbf{NCAQ}_n \to \mathbf{SMfld}_n \,.$$

- We define

$$\mathbf{Sim}_n \colon \mathbf{NCAQ}_n \to \mathbf{PMfld}_n$$

  similarly to our previous examples for $n = 2$ and $n = 3$.
- We define $\mathrm{In}(\mathbf{A})$ to consist of all entries in noncommuting tuples of $\mathbf{A}$ and $\mathrm{Out}(\mathbf{A})$ to consist of all $f(a_1, \ldots, a_n)$ where $(a_1, \ldots, a_n) \in \mathrm{NCT}(\mathbf{A})$.

The first functor: Open serenation

- We set

$$\mathsf{Sim}(\mathbf{A}) \coloneqq \{\, \underline{a} \mid a \in \mathsf{In}(\mathbf{A}) \,\} \cup \{\, \overline{a} \mid a \in \mathsf{Out}(\mathbf{A}) \,\}$$

  and

$$\mathsf{SimFace}(\mathbf{A}) \coloneqq \bigcup_{a \in \mathsf{NCT}(\mathbf{A})} \mathsf{Sb}\left(\left\{\underline{a}_1, \ldots, \underline{a}_n, \overline{f(a)}\right\}\right).$$

- We define

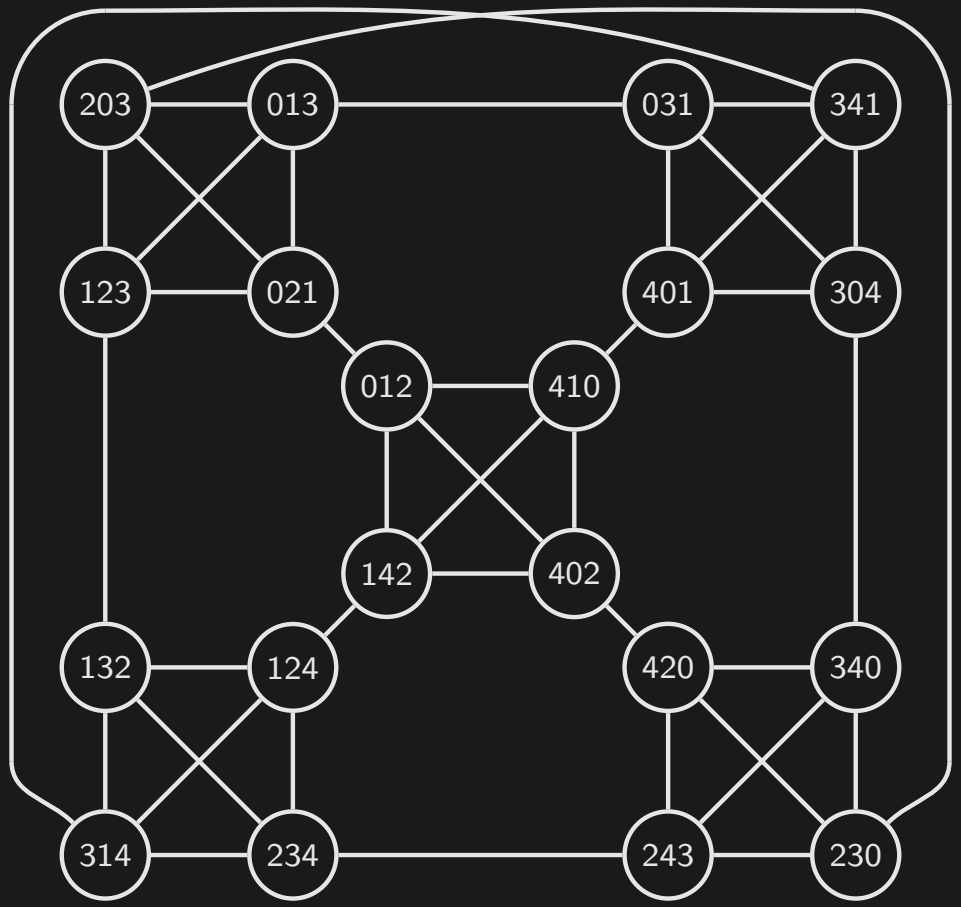
$$\mathbf{Sim}_n(\mathbf{A}) \coloneqq (\mathsf{Sim}(\mathbf{A}), \mathsf{SimFace}(\mathbf{A})).$$

The first functor: Open serenation

- We create $\mathbf{OSer}_n(\mathbf{A})$ by taking the open geometric realization of $\mathbf{Sim}_n(\mathbf{A})$ (basically all but the $(n-2)$-skeleton of the geometric realization) and then equipping it with a smooth atlas.

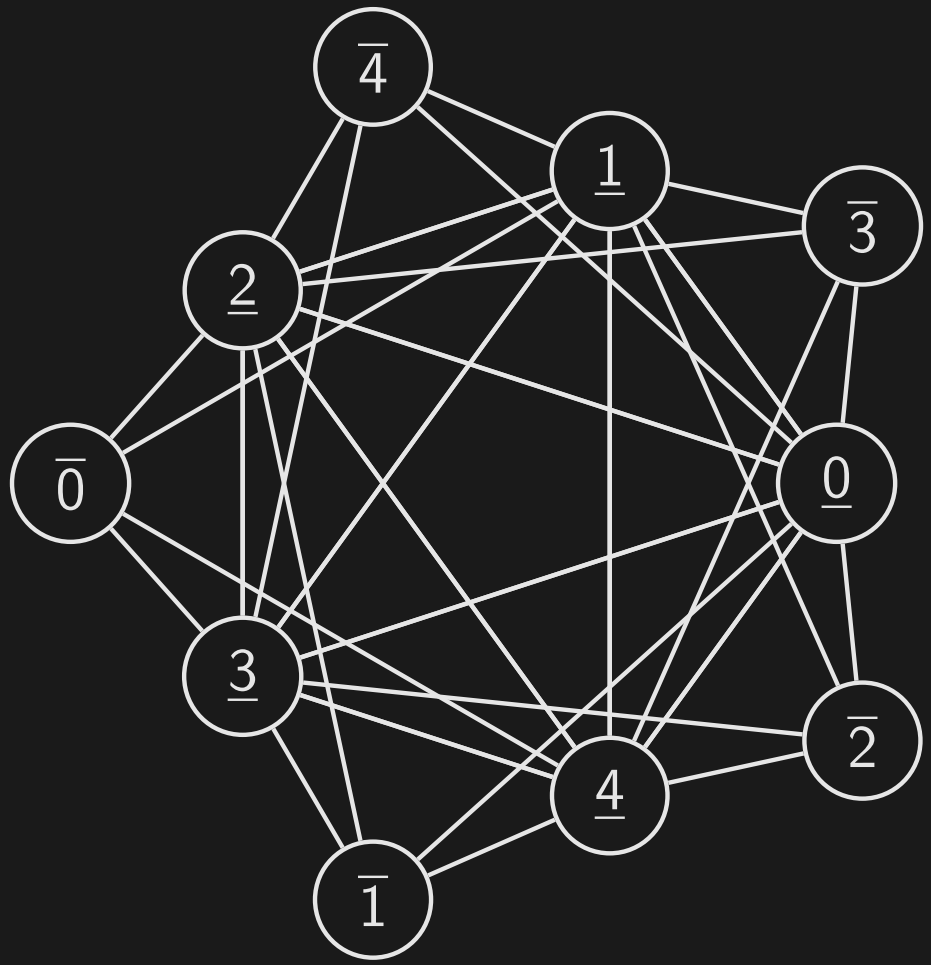
The first functor: Open serenation

- The incidence graph of the facets of **Sim**(**A**) for the ternary quasigroup **A** from the previous example is pictured.



The first functor: Open serenation

- The 1-skeleton of **Sim**(**A**) for the ternary quasigroup **A** from the previous example is pictured.



The first functor: Open serenation

- One may verify that $\mathbf{OSer}(\mathbf{A})$ is a 3-sphere minus the graph pictured previously, which is homotopy equivalent to the wedge sum of 21 circles.

The second functor: Serenation

- For any alternating quasigroup $\mathbf{A}$ we may equip $\mathbf{OSer}(\mathbf{A})$ with a Riemannian metric in a functorial manner which makes $\mathbf{OSer}(\mathbf{A})$ flat.
- We then define a *Euclidean metric completion functor*

$$\mathbf{EuCmplt}\colon \mathbf{Riem}_n \to \mathbf{Mfld}_n$$

which assigns to a Riemannian manifold $(\mathbf{M}, g)$ the topological manifold consisting of all points in the metric completion of $\mathbf{M}$ which are locally Euclidean.

The second functor: Serenation

- The *serenation functor*

$$\mathbf{Ser}_n \colon \mathbf{NCAQ}_n \to \mathbf{Mfld}_n$$

  is given by

$$\mathbf{Ser}(\mathbf{A}) := \mathbf{EuCmplt}(\mathbf{OSer}(\mathbf{A}), g)$$

  where g is the standard metric on $\mathbf{OSer}(\mathbf{A})$.
- In the previous example of the ternary quasigroup $\mathbf{A}$ we find that $\mathbf{Ser}_3(\mathbf{A})$ is the 3-sphere.

The second functor: Serenation

Definition (Serene manifold)

We say that a connected orientable n-manifold $\mathbf{M}$ is *serene* when there exists some alternating n-quasigroup $\mathbf{A}$ such that $\mathbf{M}$ is a component of $\mathbf{Ser}(\mathbf{A})$.

The second functor: Serenation

Theorem (A., Yoo (2021))

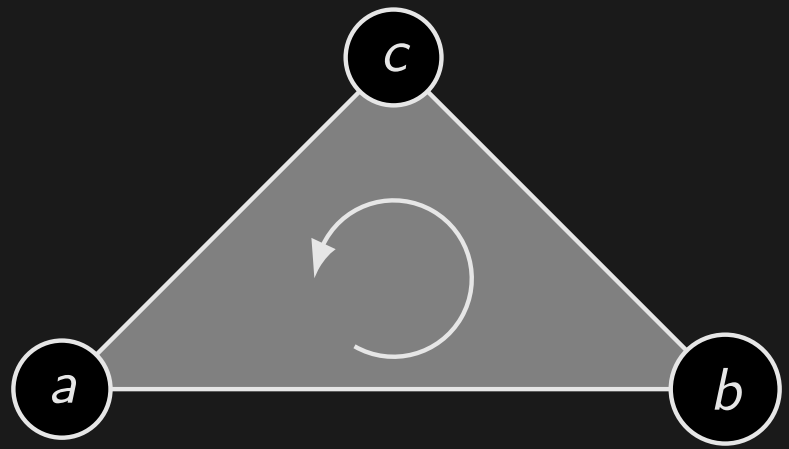
Every connected orientable triangulable n-manifold is serene.

The second functor: Serenation

Theorem (A., Yoo (2021))

Every connected orientable triangulable n-manifold is serene.

- We will give a proof by pictures in the dimension 2 case.
- Suppose that **M** is such a 2-manifold with a fixed triangulation and compatible orientation.

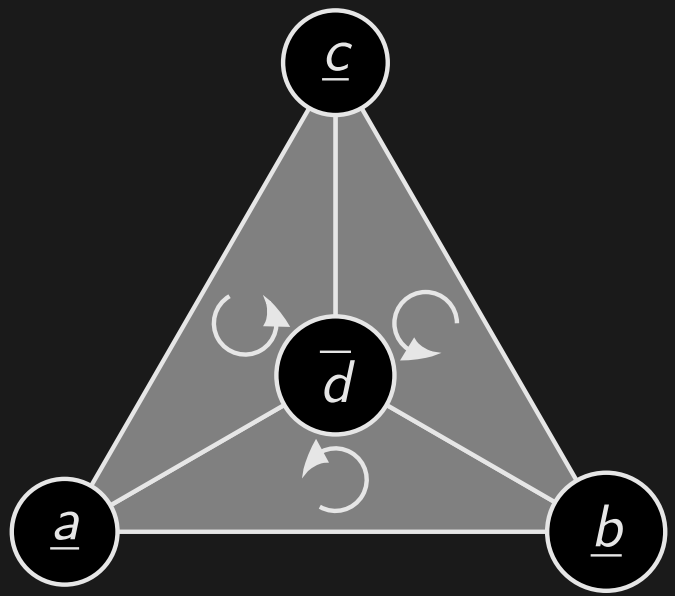


The second functor: Serenation

Theorem (A., Yoo (2021))

Every connected orientable triangulable n-manifold is serene.

- Perform the elementary subdivision of each facet of $\mathbf{M}$.

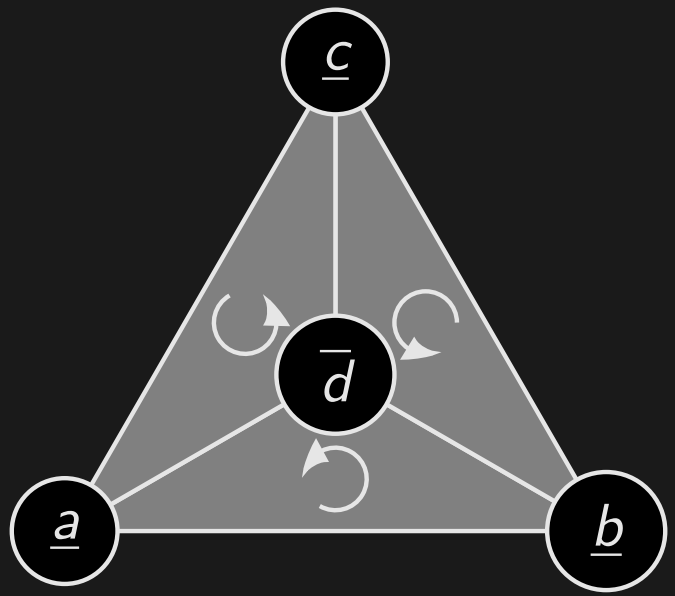


The second functor: Serenation

Theorem (A., Yoo (2021))

Every connected orientable triangulable n-manifold is serene.

- The appropriate choice of alternating n-quasigroup $\mathbf{A}$ is given a presentation with generators including $\{a, b, c, d\}$ and relations $d = f(a, b) = f(b, c) = f(c, a)$.



The Evans Conjecture and Latin cubes

Definition (Partial Latin cube)

Given a set A and some $n \in \mathbb{N}$ we say that $\theta \subset A^{n+1}$ is a *partial Latin n-cube* when for each $i \in [n]$ and each

$$a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_{n+1} \in A^n$$

there exists at most one $a_i \in A$ so that

$$(a_1, \ldots, a_{n+1}) \in \theta.$$

The Evans Conjecture and Latin cubes

- Evans conjectured that each partial Latin square (i.e. a partial Latin cube $\theta \subset A^{2+1}$) with $|A| = k$ and $|\theta| \leq k - 1$ could be filled in so as to obtain a complete Latin square $\psi \subset A^3$ with $\theta \subset \psi$ and $|\psi| = k^2$.
- This was proven to be true by Smetaniuk in 1981.
- Similar results are known for special classes of higher-dimensional Latin cubes.

The Evans Conjecture and Latin cubes

- In general a *complete Latin n-cube* is the graph of an n-quasigroup operation.
- We say that a partial Latin n-cube is *alternating* when we have for each $\alpha \in \mathrm{Alt}_n$ that if

$$(a_1, \ldots, a_n, b_1) \in \theta$$

and

$$(a_{\alpha(1)}, \ldots, a_{\alpha(n)}, b_2) \in \theta$$

then $b_1 = b_2$.

- Given a finite partial alternating Latin cube $\theta \subset A^{n+1}$ does there always exist a finite complete alternating Latin cube $\psi \subset B^{n+1}$ such that $\theta \subset \psi$?

Thank you!