Examples for the Fundamental Theorem of Algebraic PCSP

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Introduction

- These notes refer to the categorical treatment of PCSPs as described in: Maximilian Hadek, Tomáš Jakl, and Jakub Opršal. “A categorical perspective on constraint satisfaction: The wonderland of adjunctions.” In: *arXiv e-prints* (Mar. 2025). arXiv: 2503.10353 [cs.LO]

Introduction

- Reductions
- Fundamental theorem
- Gadgets
- Minion homomorphisms

Reductions

Definition (Promise (decision) problem)

A *promise (decision) problem* over some class C of elements, called *instances* is a pair of subclasses $Y, N \subset C$. We refer to members of Y as *YES instances* and members of N as *NO instances*.

- We say that a promise problem is *well-defined* when Y and N are disjoint. We usually assume this.
- We may write (C, Y, N) to denote a promise problem.

Reductions

Definition (Reduction)

A *reduction* from a promise problem (C, Y, N) to (C', Y', N') is a function $f\colon C \to C'$ such that $f(Y) \subset Y'$ and $f(N) \subset N'$.

- We will usually focus on *tractable* promise problems, which are those that have polynomial-time algorithm for deciding whether a given instance is in the YES or NO class.
- We are therefore concerned with *efficient* reductions, which are those that can be computed in polynomial time, as these preserve tractability.

Reductions

- Given objects A and B of a category $\mathscr{C}$, we denote by $A \to B$ the existence of a morphism from A to B in $\mathscr{C}$.
- Similarly, we denote by $A \not\to B$ the absence of such a morphism.

Reductions

Definition (Promise CSP)

Let A and B be objects in the category $\mathscr{C}$. A *promise CSP* for the template (A, B) is is the promise decision problem whose instances are $\mathrm{Ob}(\mathscr{C})$, whose `YES` instances are objects I with $I \to A$, and whose `NO` instances are object I with $I \not\to B$.

- We might write $\mathrm{PCSP}_{\mathscr{C}}(A, B) = (\mathrm{Ob}(\mathscr{C}), Y(A), N(B))$ to indicate this promise decision problem.
- We might write $\mathrm{PCSP}(A, B)$ as a shorthand when the category $\mathscr{C}$ is clear from context.
- We usually assume that $A \to B$.

Reductions

- Let $\mathrm{Ar}(\mathscr{C})$ denote the category of arrows for the category $\mathscr{C}$.
- Let $f\colon A \to B$ be a morphism in $\mathscr{C}$.
- The map $\psi\colon \mathrm{Ob}(\mathscr{C}) \to \mathrm{Ob}(\mathrm{Ar}(\mathscr{C}))$ given by $\psi(X) = 1_X$ is a reduction from $\mathrm{PCSP}(A, B)$ to $\mathrm{PCSP}(f, f) = \mathrm{CSP}(f)$.
- If I is a `YES` instance with $g\colon I \to A$, then 1_I is a `YES` instance since $(g, f \circ g)\colon 1_I \to f$.
- If I is a `NO` instance then certainly there cannot be a morphism from 1_I to f in $\mathrm{Ar}(\mathscr{C})$, as this would mean that $I \to B$.

Reductions

- Note that even for a fixed choice of A and B there could be many such reductions, one for each $f\colon A \to B$.
- Are these efficient? What do they tell us about the original $\mathrm{PCSP}(A, B)$?

Fundamental theorem

Theorem (Fundamental theorem of the algebraic approach to PCSPs)

Let (A, A') *and* (B, B') *be promise templates over the categories* $[\mathcal{S}^{\mathrm{op}}, \mathrm{Fin}]$ *and* $[\mathcal{T}^{\mathrm{op}}, \mathrm{Fin}]$*, respectively. The following are equivalent:*

1. *There is a gadget* $G\colon \mathcal{S} \to [\mathcal{T}^{\mathrm{op}}, \mathrm{Fin}]$ *with* $\widehat{G}$ *a log-space reduction from* $\mathrm{PCSP}(A, A')$ *to* $\mathrm{PCSP}(B, B')$*.*
2. *There is a gadget* G *and homomorphisms* $A \to N_G B$ *and* $N_G B' \to A'$*.*
3. *There are functors* $L \dashv R$ *with* $A \to RB$ *and* $RB' \to A'$*.*
4. *There is a natural transformation* $\mathrm{Pol}(B, B') \to \mathrm{Pol}(A, A')$*.*

Gadgets

- We give a gadget $G\colon \mathcal{S} \to [\mathcal{T}^{\mathrm{op}}, \mathrm{Fin}]$ which realizes our reduction ψ from before.
- We'll do this in the case where $[\mathcal{S}^{\mathrm{op}}, \mathrm{Fin}]$ is the category of graphs, but the same kind of construction should work in general.

Gadgets

- A combinatorial graph may be thought of as a set V of vertices, a set E of edges, and a pair of maps $s\colon E \to V$ and $t\colon E \to V$ where $s(e)$ and $t(e)$ are the source and target of the directed edge e.
- This is a contravariant functor from

  $$V \overset{s}{\underset{t}{\rightrightarrows}} E$$

  to the category Fin.

Gadgets

- We take the category $\mathcal{T}$ to be the "doubling" of $\mathcal{S}$ given by

$$\begin{tikzcd} V_1 \arrow[r, shift left, "s_1"] \arrow[r, shift right, "t_1"'] & E_1 \\ V_2 \arrow[u, "h_V"] \arrow[r, shift left, "s_2"] \arrow[r, shift right, "t_2"'] & E_2 \arrow[u, "h_E"'] \end{tikzcd}$$

  where $s_1 \circ h_V = h_E \circ s_2$ and $t_1 \circ h_V = h_E \circ t_2$.
- This is the diagram of a graph homomorphism (with arrows reversed), so $[\mathcal{T}^{\mathrm{op}}, \mathrm{Fin}] \cong \mathrm{Ar}([\mathcal{S}^{\mathrm{op}}, \mathrm{Fin}])$.

Gadgets

- The gadget $G\colon \mathcal{S} \to [\mathcal{T}^{\mathrm{op}}, \mathrm{Fin}]$ is given by setting $G(V)$ to be the identity morphism on a graph with a single vertex v and no edges, $G(E)$ to be the identity morphism on the graph $(\{v_s, v_t\}, \{(v_s, v_t)\})$, and taking $G(s)$ and $G(t)$ to map v to v_s and v_t, respectively.

Gadgets

- The nerve $N_G\colon [\mathcal{T}^{\mathrm{op}}, \mathrm{Fin}] \to [\mathcal{S}^{\mathrm{op}}, \mathrm{Fin}]$ is given by

$$(N_G(h\colon B_1 \to B_2))(V) = [\mathcal{T}^{\mathrm{op}}, \mathrm{Fin}](G(V), h) \cong B_1(V),$$

$$(N_G(h\colon B_1 \to B_2))(E) = [\mathcal{T}^{\mathrm{op}}, \mathrm{Fin}](G(E), h) \cong B_1(E),$$

  and $(N_G(h))(s)$ and $(N_G(h))(t)$ behave as the expected source and target maps from $B_1(E)$ to $B_1(V)$.
- This is all to say that $N_G(h\colon B_1 \to B_2) = B_1$, so N_G picks out the domain of a homomorphism.

Gadgets

- Let's go back to our claim that $\psi(X) = 1_X$ is a reduction from $\mathrm{PCSP}(A, B)$ where $f \colon A \to B$ to $\mathrm{PCSP}(f, f) = \mathrm{CSP}(f)$.
- There is a gadget G of the appropriate type, so we must show that $A \to N_G(f)$ and $N_G(f) \to B$.
- This amounts to showing that $A \to A$ and $A \to B$, so we can take $1_A \colon A \to A$ and $f \colon A \to B$ to see that the theorem applies to this situation.
- This means that $\widehat{G} \dashv N_G$ is a log-space reduction from $\mathrm{PCSP}(A, B)$ to $\mathrm{CSP}(f)$.

Gadgets

- For a graph X and a graph homomorphism $h\colon Y_1 \to Y_2$ we have that

$$[\mathcal{T}^{\mathrm{op}}, \mathrm{Fin}](\widehat{G}(X), h) \cong [\mathcal{S}^{\mathrm{op}}, \mathrm{Fin}](X, N_G(h))$$

  so

$$[\mathcal{T}^{\mathrm{op}}, \mathrm{Fin}](\widehat{G}(X), h) \cong [\mathcal{S}^{\mathrm{op}}, \mathrm{Fin}](X, Y_1).$$

- If we take $\widehat{G}(X) = 1_X$ then this will be the case, as in our initial definition of $\psi(X) = 1_X$.

Minion homomorphisms

- It is natural to wonder whether there is a reduction from $\mathrm{CSP}(f)$ to $\mathrm{PCSP}(A, B)$, since $f \colon A \to B$ appears to contain a lot of information about A and B.
- Another part of the fundamental theorem shows us that this can't be expected in general.

Minion homomorphisms

- Since we know that there is a reduction from $\mathrm{PCSP}(A, B)$ to $\mathrm{PCSP}(f, f)$, this means that there is a natural transformation from $\mathrm{Pol}(f, f)$ to $\mathrm{Pol}(A, B)$.
- Note that

$$(\mathrm{Pol}(A, B))(n) = [\mathcal{S}^{\mathrm{op}}, \mathrm{Fin}](A^n, B)$$

while

$$\begin{aligned}
&(\mathrm{Pol}(f, f))(n) = \\
&[\mathcal{T}^{\mathrm{op}}, \mathrm{Fin}](f^n, f) = \\
&\{\, (\sigma, \tau) \in (\mathrm{Pol}(A))(n) \times (\mathrm{Pol}(B))(n) \mid f \circ \sigma = \tau \circ f^n \,\}\,.
\end{aligned}$$

Minion homomorphisms

- The natural transformation from $\mathrm{Pol}(f, f)$ to $\mathrm{Pol}(A, B)$ is given by $\eta_n(\sigma, \tau) = f \circ \sigma$.
- It is possible that there are many morphisms $A^n \to B$ even when $\mathrm{Pol}(A)$ and $\mathrm{Pol}(B)$ are very small or have very few members related by f in this way, so we can't always hope for a natural transformation back in the other direction.