The Topology of Magmas

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Magmas

Definition (Magma)

A *magma* (or *binar* or, classically, *groupoid*) is an algebraic structure (S, f) consisting of an underlying set S and a single binary operation $f\colon S^2 \to S$.

Operation Digraphs

Definition (Operation digraph)

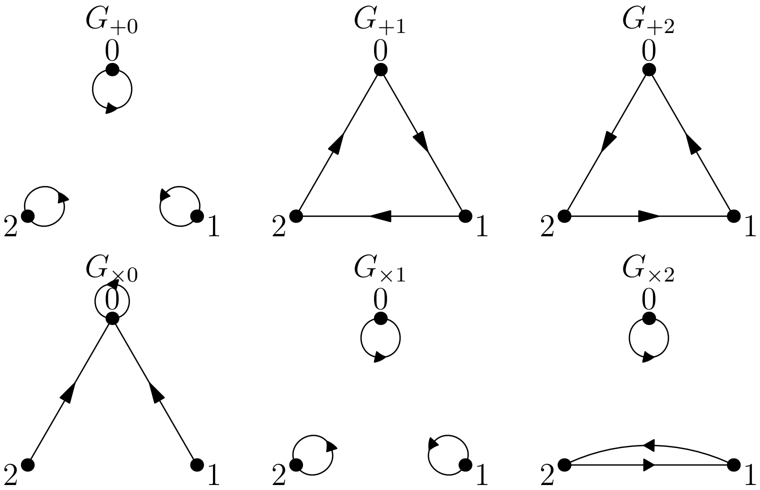
Let $f\colon S \to S$ be a unary operation. The *operation digraph* (or *functional digraph*) of f, written G_f, is given by $G_f = G(S, E)$ where

$$E = \{(s, f(s)) \mid s \in S\}.$$

Definition (Operation digraph for a binary operation)

Let $f\colon S^2 \to S$ be a binary operation and let $s \in S$. The *left operation digraph* of s under f, written G^L_{fs}, is the operation digraph of $f^L_s\colon S \to S$ where $f^L_s(x) := f(s, x)$ for $x \in S$. The *right operation digraph* of s under f, written G^R_{fs}, is defined analogously.

Example: Operation Digraphs from $\mathbb{Z}/3\mathbb{Z}$



Previous Work in...

- Semigroup theory
- Dynamics and number theory
- Cayley graphs
- Graph theory
- Universal algebra (unary algebras)

Operation Matrices

Definition (Adjacency matrix)

Let $G(V, E)$ be a digraph, let $|V| = n$, and fix an order on the vertex set V. The *adjacency matrix* A for G under the given order on V is the $n \times n$ matrix whose ij-entry is 1 if there is an edge in G from v_i to v_j and 0 otherwise.

We write A^L_{fs} to indicate the adjacency matrix of G^L_{fs} and similarly write A^R_{fs} to indicate the adjacency matrix of G^R_{fs}.

Example: Operation Matrices from $\mathbb{Z}/3\mathbb{Z}$

$$A_{+0} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad A_{+1} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \quad A_{+2} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$A_{\times 0} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad A_{\times 1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad A_{\times 2} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Example: Operation Matrices from $\mathbb{Z}/3\mathbb{Z}$

Write s_i to indicate i viewed as an element of $\mathbb{Z}/3\mathbb{Z}$. Multiplying a vector by the adjacency matrix of an operation digraph corresponds to applying the corresponding function to the corresponding element.

$$s_2 A_{+1} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} = s_0$$

$$s_1 A_{+2} = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} = s_0$$

Graph Treks

Theorem

Let A be the adjacency matrix for G with a given vertex ordering. Then $(A^k)_{ij}$ for $k \in \mathbb{N}$ is the number walks of length k from v_i to v_j in G.

It is natural to consider the significance of the product of the adjacency matrices of two or more different graphs on the same set of vertices.

Graph Treks

Definition (Trek)

Let $(G_1, G_2, \ldots, G_k)$ be a tuple of graphs on a common set of vertices V. A *trek* (or (v_i, v_j)-*trek*) on $(G_1, G_2, \ldots, G_k)$ is an ordered list of vertices and edges $v_i, e_1, \ldots, e_k, v_j$ where $e_t \in E(G_t)$ is an edge joining the vertices before and after it in the list.

Theorem (A. 2015)

Let $(G_1, G_2, \ldots, G_k)$ be a tuple of graphs on a set of vertices V under a given vertex ordering and let $A_1, A_2, \ldots, A_k$ be the corresponding adjacency matrices. Then $(A_1 A_2 \cdots A_k)_{ij}$ is the number of treks on $(G_1, G_2, \ldots, G_k)$ of length k from v_i to v_j.

Counting Solutions to Equations

Multiplying operation matrices corresponds to function composition:

$$A_{\times 2} A_{+1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

This also corresponds to looking at those treks which consist of a step on $G_{\times 2}$ followed by a step on G_{+1}.

Counting Solutions to Equations

Theorem (A. 2015)

Let S be an ordered finite set of elements and let $\{f_p\}_{p \in P}$ where $f_p \colon S \to S$ be an indexed collection of functions. Let $G_p = G(S, E_p)$ be the operation digraph for f_p and let A_p be the adjacency matrix for G_p under the given ordering for S. If $Q = \{q_n\}_{n=1}^{k}$ is a finite sequence of k elements of P and $y = s_j$ is a fixed element of S we have that the number of $x \in S$ for which $f^Q(x) = y$ is exactly $\sum_{i=1}^{|S|} \left(\prod_{n=1}^{k} (A_{q_n}) \right)_{ij}$.

Counting Solutions to Equations

Theorem (Sylvester's Rank Inequality)

Let U, V, and W be finite-dimensional vector spaces, let A be a linear transformation from U to V and let B be a linear transformation from V to W. Then $\operatorname{rank} BA \geq \operatorname{rank} A + \operatorname{rank} B - \dim V$.

By induction we see that for a finite collection of linear transformations $\{A_i \colon V \to V\}_{i \in I}$ we have $\operatorname{rank} \prod_{i \in I} A_i \geq \left(\sum_{i \in I} \operatorname{rank} A_i\right) - (|I| - 1) \dim V$.

Example: An Equation over $\mathbb{Z}/4\mathbb{Z}$

$$((3(x+2))^3)^{((3(x+2))^3)} = y$$

Let $f_1(x) = x + 2$, $f_2(x) = 3x$, $f_3(x) = x^3$, and $f_4(x) = x^x$. Note that the equation under consideration can be rewritten as $f^Q(x) = y$, where Q is the sequence $(1, 2, 3, 4)$.

$$A_1 = A_{+2} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \quad A_2 = A_{\times 3} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$A_3 = A^R_{\wedge 3} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad A_4 = A^R_{\uparrow 2} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Example: An Equation over $\mathbb{Z}/4\mathbb{Z}$

$$\left((3(x+2))^3\right)^{\left((3(x+2))^3\right)} = y$$

Since $\operatorname{rank} A_{+2} = \operatorname{rank} A_{\times 3} = 4$ and $\operatorname{rank} A^R_{\wedge 3} = \operatorname{rank} A^R_{\uparrow 2} = 3$, we have that

$$\begin{aligned}\operatorname{rank} \prod_{n=1}^{4} A_n &\geq \left(\sum_{n=1}^{4} \operatorname{rank} A_n\right) - (|I| - 1)|S| \\ &= (4+4+3+3) - (4-1)4 \\ &= 2.\end{aligned}$$

Sylvester's Inequality for Functions

Proposition (Sylvester's inequality for functions)

Let X, Y, and Z be finite sets and let $f : X \to Y$ and $g : Y \to Z$ be functions. Then

$$|(g \circ f)(X)| \geq |f(X)| + |g(Y)| - |Y|.$$

Operation Hypergraphs

Definition (Operation hypergraph)

Let $f\colon S^2 \to S$ be a binary operation. The *operation hypergraph* of f, written G_f, is given by $G_f = G(S, E)$ where

$$E = \{(s_i, s_j, f(s_i, s_j)) \mid s_i, s_j \in S\}.$$

Adjacency Tensor

Definition (Adjacency tensor)

Let $G(V,E)$ be a 3-uniform hypergraph, let $|V| = n$, and fix an order on the vertex set V. The *adjacency tensor* A for G under the given order on V is the $n \times n \times n$ hypermatrix whose ijk-entry is 1 if (v_i, v_j, v_k) is an edge in G and 0 otherwise.

Recall that given such a tensor we can obtain a bilinear map $A_f : \mathbb{C}^S \times \mathbb{C}^S \to \mathbb{C}^S$ where given $x_1 = (a_s s)_{s \in S}$ and $x_2 = (b_s s)_{s \in S}$ from $\mathbb{R}^S$ we define

$$A_f(x_1, x_2) := \sum_{s_i, s_j, s_k \in S} a_{s_i} b_{s_j} (A_f)_{ijk} s_k = \sum_{s_i, s_j \in S} a_{s_i} b_{s_j} f(s_i, s_j).$$

Hypergraph Odysseys

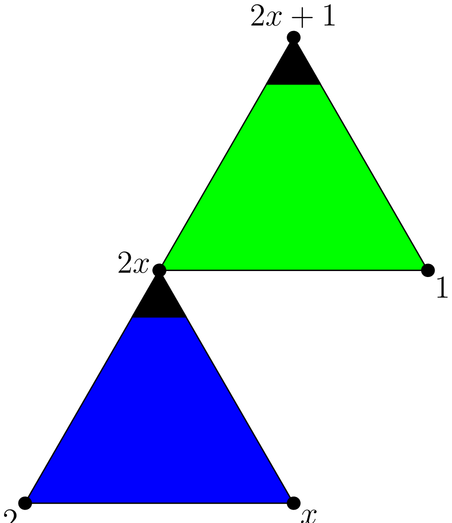
There are many ways to compose binary operations. Let $f, g\colon S^2 \to S$.

$$(x, y, z) \mapsto g(f(x, y), z)$$

$$(x, y, z) \mapsto f(f(x, x), g(x, f(x, f(y, z)))).$$

Hypergraph Odysseys

We return to our $2x + 1 = y$ example.

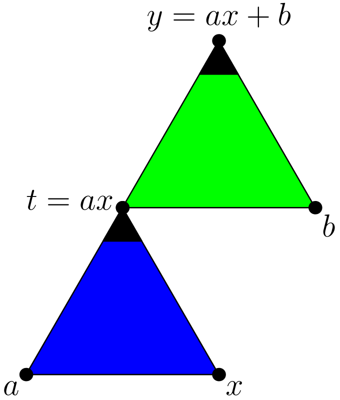


Hypergraph Odysseys

Definition (μ, Σ-odyssey)

Let X and Y be sets of variables and take Σ to be a collection of pairs of the form (e, E) where $E = E_i$ for some $i \in I$ and $e \in (X \uplus Y)^{\rho(i)}$. If there exist evaluation maps $\mu\colon X \to S$ (the *endpoint evaluation map*) and $\nu\colon Y \to S$ (the *intermediate point evaluation map*) such that for each $(e, E) \in \Sigma$ we have that $(\mu \circ \nu)(e) \in E$ then we say that the collection of edges $\mathscr{O} = (\mu \circ \nu)(e)$ is a Σ-*odyssey* on the G_i. We say that X is the set of *end variables*, Y is the set of *intermediate variables*, $\mu(X)$ is the set of *endpoints*, $\nu(Y)$ is the set of *intermediate points*, Σ is the *odyssey type*, and $|\Sigma|$ is the *length* of the odyssey. We call a Σ-odyssey $\mathscr{O}$ a μ, Σ-*odyssey* if $\mu\colon X \to S$ is the endpoint evaluation map of $\mathscr{O}$ for some fixed μ.

Hypergraph Odysseys



End variables: $X = \{x, y, a, b\}$
Intermediate variable: $Y = \{t\}$
Odyssey type: $\Sigma = \{((a, x, t), G_\times), ((t, b, y), G_+)\}$

Counting Solutions to Equations

Let φ denote the logical formula

$$\varphi(a, b, x, y) := (\exists t \in \mathbb{Z}/3\mathbb{Z})((a, x, t) \in G_\times \wedge (t, b, y) \in G_+).$$

Let A and B be arbitrary rank 3 tensors over $\mathbb{C}$. Define

$$(\varphi AB)_{ijkl} := \sum_{t \in \{0,1,2\}} A_{ikt} B_{tjl},$$

which is the *generalized matrix product* of A and B corresponding to the logical formula φ. By simple definition-chasing one finds that $\varphi G_\times G_+$ is the adjacency tensor for the composite operation

$$(a, b, x) \mapsto ax + b.$$

Embedding Dimension

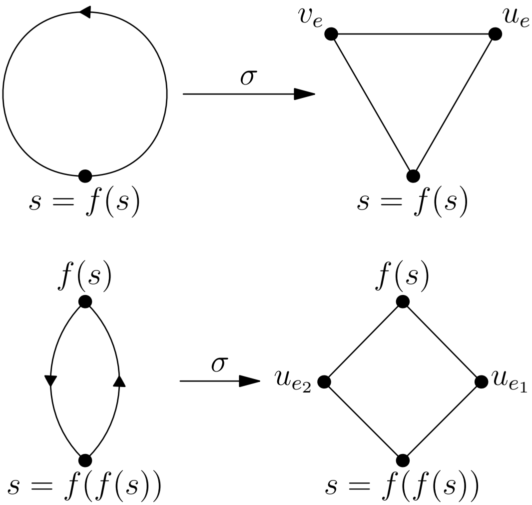
Definition (Operation graph)

Let $f\colon S \to S$ be a unary operation. The *operation graph* of f, written $\bar{G}_f$, is the simple graph $G(V, E)$ which is constructed as follows. For each edge $e = (s, f(s))$ in G_f define

$$\sigma(e) := \begin{cases} \{(s, u_e), (u_e, v_e), (v_e, s)\} & \text{when } f(s) = s \\ \{(s, u_e), (u_e, f(s))\} & \text{when } f^2(s) = s \text{ and } f(s) \neq s \\ \{e\} & \text{otherwise} \end{cases}$$

where u_e and v_e are new vertices unique to the edge e. Take $E = \bigcup_{e \in E(G_f)} \sigma(e)$ and let V be the union of S and all the u_e and v_e generated by applying σ to edges $e \in E(G_f)$.

Embedding Dimension



Embedding Dimension

Theorem

Every operation graph is planar.

Theorem

Let H be a subdivision of a simple graph H' with n vertices, each of degree at least $k+1$ for $k \geq 2$. The graph H cannot appear as a subgraph of any operation graph if $k > \frac{n-1}{2}$.

Embedding Dimension

Definition (Operation complex)

Let $f\colon S^2 \to S$ be a binary operation. The *operation complex* of f, written $\bar{G}_f$, is the simplicial complex whose 2-faces are the edges of the hypergraph $G(V,E)$, which is constructed as follows. Write $(a,b,c,d)_2$ to indicate the set of all 2-faces of the simplex with vertices a, b, c, and d. For each edge $e = (s_i, s_j, f(s_i,s_j))$ in G_f define

$$\sigma(e) := \begin{cases} (s_i, u_e, v_e, w_e)_2 & \text{when } |\{s_i, s_j, f(s_i,s_j)\}| = 1 \\ (s_i, s_j, u_e, v_e)_2 & \text{when } |\{s_i, s_j, f(s_i,s_j)\}| = 2 \\ (s_i, s_j, s_k, u_e)_2 & \text{when } |\{s_i, s_j, f(s_i,s_j)\}| = 3 \text{ and } \tau e \in f \text{ for} \\ & \text{some nonidentity permutation } \tau \\ \{e\} & \text{otherwise} \end{cases}$$

where u_e, v_e, and w_e are new vertices unique to the edge e. Take $E = \bigcup_{e \in E(G_f)} \sigma(e)$ and let V be the union of S and all the u_e, v_e, and w_e generated by applying σ to edges $e \in E(G_f)$.

Embedding Dimension

Given any magma (S, f) we then know that $\bar{G}_f$ embeds into $\mathbb{R}^k$ but not $\mathbb{R}^{k-1}$ for some $k \in \{3, 4, 5\}$.

Definition (Embedding dimension)

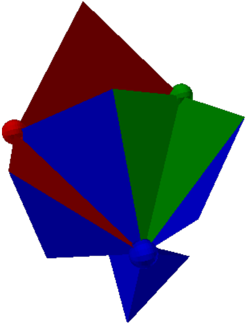
Let (S, f) be a magma with operation complex $\bar{G}_f$. We refer to the minimal k such that the complex $\bar{G}_f$ embeds into $\mathbb{R}^k$ as the *embedding dimension* of the magma (S, f).

The situation here is more complex than for unary operations.

Embedding Dimension

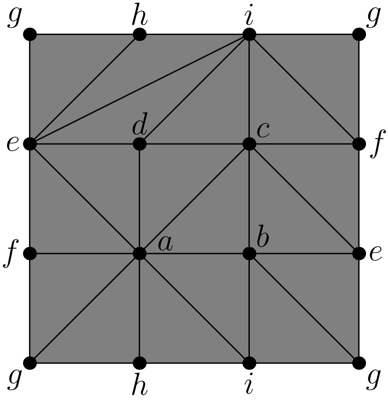
Let (S, f) be a magma such that for every $x, y \in S$, $x \neq y$, we have that either $f(x, y) = x$ or $f(x, y) = y$. Every edge $e \in G_f$ then contains at most 2 vertices which belong to S. We can embed $\bar{G}_f$ into $\mathbb{R}^3$ without self-intersections.

There are also magmas of embedding dimension 3 without this property. Consider $(\mathbb{Z}_3, +)$.



Embedding Dimension

Consider the triangulation $\mathrm{Kh}12$ of the Klein bottle.



Embedding Dimension

We orient faces to obtain a partial operation table.

	a	b	c	d	e	f	g	h	i
a	·	c	d	e	f	g	h	i	b
b	·	·	·	·	·	·	·	·	·
c	·	·	·	i	b	·	·	·	·
d	·	·	·	·	i	·	·	·	·
e	·	·	·	·	·	c	b	·	·
f	·	·	·	·	·	·	i	·	c
g	·	·	·	·	h	·	·	·	b
h	·	·	·	·	·	·	·	·	e
i	·	·	·	·	·	·	·	·	·

This "forbidden substructure" cannot appear in any magma with embedding dimension 3.

Embedding Dimension

- If a magma has embedding dimension n then clearly every submagma has embedding dimension at most n. How does embedding dimension behave under taking products or homomorphic images of magmas?
- If the class "magmas of embedding dimension at most n" is closed under taking homomorphic images, submagmas, and products we would have an equational class (Birkhoff's Variety Theorem).

Spectrum Calculation

Theorem

Let $f : S \to S$ be a function on a set S of size n. Let $m(j)$ denote the number of j-cycles under f and let Z_j denote the multiset which consists of $m(j)$ copies of each j^{th} root of unity. The nonzero part of the spectrum of A_f is the multiset union $\bigcup_j Z_j$.

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